CAVITATION IN NONLINEAR ELASTICITY
AND ASSOCIATED PROBLEMS

by

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ABSTRACT

We study a class of weak solutions to the equilibrium equations of nonlinear elasticity in which a hole forms at the centre of a ball in tension. It is shown that there exists a critical value of the boundary displacement at which a stable solution, corresponding to a deformation with a cavity, bifurcates from the homogeneous solution (which then loses stability). This part of our analysis extends work of Ball (1982) to more general stored energy functions.

In Chapter 2 the asymptotic behaviour of the critical loads and displacements at which bifurcation occurs is determined in the incompressible limit.

We approximate the singular problem of cavitation in a solid ball by means of a sequence of non singular problems for punctured balls B^{ε} of internal radius ε. Using phase plane arguments we prove the uniqueness of solutions to boundary value problems for B^{ε} and are then led to a proof of uniqueness of solutions with cavities for the solid ball. The asymptotic behaviour of the punctured ball solutions as $\varepsilon \longrightarrow 0$ is determined and a uniform first order expansion constructed.

Finally we interpret the phenomonon of cavitation using elements of the field theory of the calculus of variations.

INTRODUCTION

Consider a homogeneous ball of isotropic elastic material which in its reference configuration occupies the region $B = \{\underline{X} \in \mathbb{R}^3; |\underline{X}| < 1\}$. A deformation of the ball corresponds to a function $\underline{x} : B \longrightarrow \mathbb{R}^3$ and in hyperelasticity we associate with $\underline{x}$ an energy E which is given by

$$E(\underline{x}) = \int_B W(\nabla \underline{x}(\underline{X}))\,dX \ , \tag{1}$$

where W is the stored energy function of the material. The equilibrium equations of non linear elasticity under zero body force are the Euler-Lagrange equations for (1), that is

$$\frac{\partial}{\partial X^\alpha}\left[\frac{\partial W}{\partial x^i_{,\alpha}}(\nabla \underline{x}(\underline{X}))\right] = 0 \quad i = 1,2,3 \tag{2}$$

and $\underline{x}$ is said to be a weak equilibrium solution if (2) holds in the sense of distributions.

In this thesis we consider radial deformations, that is $\underline{x}$ of the form

$$\underline{x}(\underline{X}) = \frac{r(R)}{R}\,\underline{X} \ , \tag{3}$$

where $R = |\underline{X}|$. As the material is isotropic $W(F)$ may be expressed as a symmetric function Φ of the eigen-values v_i of $(F^TF)^{\frac{1}{2}}$. Ball (1982) shows that the study of weak equilibrium solutions of the form (3) is equivalent to studying solutions to the radial equilibrium equation

$$\frac{d}{dR}\left[R^2\Phi_{,1}(r'(R), \frac{r(R)}{R}, \frac{r(R)}{R})\right] = 2R\Phi_{,2}(r'(R), \frac{r(R)}{R}, \frac{r(R)}{R}), \tag{4}$$

where $\Phi_{,i}$ denotes differentiation of Φ with respect to its i^{th} argument, and he exhibits a class of solutions to (4) that satisfy $r(0) > 0$. This corresponds to a hole forming at the centre of the deformed ball and following Ball (1982) we term this cavitation. We say that a solution r of (4) is a cavitating equilibrium solution if it satisfies $r(0) > 0$ together with the natural boundary condition that the cavity surface is stress free. Ball's results however are only valid for a particular class of stored energy functions. Throughout this thesis we will treat a general form of the stored energy function rather this restricted class.

In Chapter 0 section 1 we make precise the notion of a weak equilibrium solution and in section 2 we gather together properties of solutions r to (4).

Our proof of existence of equilibrium solutions to the displacement and traction problems uses the direct method of the calculus of variations to show that a minimiser of E exists in the class of radial deformations. A key problem in the direct method is to extract a convergent subsequence from any minimising sequence and in the case of our radial problem difficulty arises from the singularity of the integrand at the origin. Ball (1982) overcomes this through a change of variables relative to which the integrand is non singular. Our approach differs in that we work away from the singularity on intervals of the form $[\partial,1]$ and then pass to the limit as $\partial \longrightarrow 0$. In propositions 1.8 and 1.9 we exhibit the existence of cavitating minimisers for stored energy functions satisfying

$$0 < M \leqslant \frac{\Phi(\lambda,\lambda,\lambda)}{\lambda^3}$$

for λ sufficiently large. In doing so we use a technique of Ball (1982) but our approach makes explicit the role of this growth condition.

In the displacement boundary value problem the solution or r of (4) satisfies $r(1) = \lambda > 0$. When combined, the results of theorem 0.14 and proposition 1.8 give the existence of a critical boundary displacement λ_{crit} with the property that for $\lambda < \lambda_{crit}$ the homogeneous deformation is the unique minimiser of the energy for $\lambda > \lambda_{crit}$ a deformation with a cavity is the unique energy minimiser. These results extend those of Ball (1982).

In Chapter 1 section 2 we indicate how the results of section 1 may be extended to inhomogeneous materials and the chapter is concluded with a calculation of the critical load for cavitation of an incompressible inhomogeneous material. We show that the critical load depends purely on the material present at the origin and moreover that it is the same as that for a homogeneous ball composed <u>entirely</u> of this material.

The Incompressible Limit

In Chapter 2 we consider a class of stored energy functions of the form

$$W^k(F) = W^{inc}(F) + f(k,\det F-1),$$

where W^{inc} is the stored energy function of an incompressible material and f is a compressibility term with the property that

$$f(k,\partial) \longrightarrow \infty \quad \text{as} \quad k \longrightarrow 0 \quad \text{if} \quad \partial \neq 0.$$

Then

$$I^k(\underline{x}) \overset{\text{def}}{=} \int_B W^k(\nabla\underline{x})\,dX$$

satisfies $I^k(\underline{x}) \longrightarrow \infty$ as $k \longrightarrow 0$ unless $\det\nabla\underline{x} = 1$ a.e.. Thus in the limit $k \longrightarrow 0$ only incompressible deformations have finite energy; we term this the <u>incompressible limit</u>. In theorem 2.5, using a penalty argument, we show that the critical displacements λ^k_{crit} satisfy

$$\lambda^k_{crit} \longrightarrow 1 \quad \text{as} \quad k \longrightarrow 0.$$

To show convergence of the critical loads P^k_{crit} to the incompressible critical load P^{inc}_{crit} poses a more difficult problem as it involves passing to a limit in the f term in which $\det F \longrightarrow 1$ and $k \longrightarrow 0$ simultaneously. We overcome this by an alternative characterisation of the critical load as the "stress at infinity" in an infinite body. This relies on the invariance of the equilibrium equations under rescaling which is such that an infinitessimal hole in a finite expanse of material behaves as a finite hole in an infinite expanse. If $P(c)$ is the stress on the boundary of the ball for a cavitating equilibrium solution with cavity of size c, then the critical load at which bifurcation occurs is the limiting value of $P(c)$ as $c \longrightarrow 0$. Under the rescaling this may be replaced by the limiting value of the stress on the outer boundary of a finite ball that contains a fixed cavity, the limit now being taken as the size of the ball tends to infinity (with the hole size remaining constant). An explicit example of convergence of the critical loads and displacements is given in Ball (1982) ex. 7.11.

Punctured Balls

Cavitating equilibrium solutions are singular three dimensional deformations in which the origin maps to the cavity surface. We approximate the solid ball B by a sequence of domains B^{ε} which correspond in the reference configuration to 'punctured' balls of internal radius ε and in doing so we remove the point of singularity.

Equilibrium solutions to the mixed displacement traction boundary value problem for a punctured ball B^{ε} correspond to solutions r_{ε} of (4) satisfying $r_{\varepsilon}(1) = \lambda > 0$ and zero stress on the inner surface (which is non linear boundary condition). A change of variables gives (4) an autonomous form and any such solution r_{ε} generates an orbit in phase space with the property that it intersects two given curves. We parametrize the set of all orbits with this property and show that an appropriate 'time map' is a strictly monotone function of the parameter. This enables us to prove uniqueness of r_{ε} in theorem 3.4. A different choice of time map yields a proof of uniqueness of solutions to the pure displacement boundary value problem for B^{ε} (see theorem 3.5).

Intuitively we expect $\{r_\varepsilon\}$ to approximate equilibrium solutions to the displacement boundary value problem for the solid ball B. In proposition 4.5 we show that r_ε is a solution to the mixed problem for B^ε if and only if it is the global minimiser of the energy. This enables us, using energy arguments, to prove the following convergence results

$$\text{if } \lambda \leqslant \lambda_{crit} \text{ then } \sup_{[\varepsilon,1]} |r_\varepsilon(R) - \lambda R| \longrightarrow 0 \text{ as } \varepsilon \longrightarrow 0 \text{ and} \tag{5}$$

$$\text{if } \lambda > \lambda_{crit} \text{ then } \sup_{[\varepsilon,1]} |r_\varepsilon(R) - r_c(R)| \longrightarrow 0 \text{ as } \varepsilon \longrightarrow 0, \tag{6}$$

where $r_c(R)$ is a cavitating equilibrium solution for the solid ball B. (The convergence (6) was first noted in Ball (1982)).

The $\{r_\varepsilon\}$ exhibit boundary layer behaviour with significant changes in strain in a neighbourhood of the cavity. In proposition 4.7 we prove the existence of solutions r_o to (4) on the exterior domain $[1,\infty)$ that satisfy zero stress at $R = 1$ and with asymptotic behaviour $\frac{r_0(R)}{R} \longrightarrow \lambda$ as $R \longrightarrow \infty$. Using this solution we construct in theorem 4.9 the following <u>uniform</u> first order expansion for the $\{r_\varepsilon\}$, namely

$$r_\varepsilon(R) = \varepsilon r_o\left(\frac{R}{\varepsilon}\right) + o(\varepsilon).$$

Thus our expansion gives a uniform estimate of the strains within the boundary layer.

In Chapter 4 section 2 we indicate the relevance of our results in studying the interactions between voids in a material.

<u>Uniqueness of the Cavitating Solution</u>

It is readily observed that when interpreted geometrically the uniqueness proofs of Chapter 3 section 1 for punctured balls rely on the fact that two distinct orbits in phase space cannot cross and thus one solution curve either lies wholly below or above any other. Motivated by this consideration we make a change of variables in the energy functional to an integration over phase plane variables relative to which the energy is a convex function. This leads to a natural proof of uniqueness of cavitating equilibrium solutions in theorem 3.14.

This contrasts with Ball who uses an ad hoc gronwall inequality together with very restrictive hypotheses to prove uniqueness. Stuart (1984) under less restrictive conditions uses a shooting argument showing that the stress on the cavity surface is a monotone function of the derivative on the boundary. Thus in Stuart's approach uniqueness is an immediate consequence of existence. Our theorem differs from the work of Ball and Stuart not only in its generality but also through the alternative view that it gives into the underlying structure of the problem.

The Field Theory of the Calculus of Variations

Our results indicate a 1-1 correspondence between solutions of the equilibrium equations and global minimisers of the energy (see for example proposition 4.5). The classical field theory of the calculus of variations gives sufficient conditions for a solution of the Euler-Lagrange equations to be a strong local minimum of the energy. In Chapter 5 section 1 we present elements of the theory and modify a result of Weierstrass to give sufficient conditions for a solution to be a global minimiser.

Though Stuart (1984) is able to show the existence and uniqueness of cavitating equilibrium solutions, he is able to conclude little as to their stability. In Chapter 4 section 2, exploiting the invariance of the equilibrium equations under rescaling, we construct a field of extremals (proposition 5.12) and using the Weierstrass theory we are able to study stability in a full neighbourhood of these equilibria in an appropriate function space. A conservation law for finite elastostatics plays a central role in this analysis. In Ball (1982) stability is a direct consequence of his variational method of proving existence once he proves uniqueness of the cavitating equilibrium solution. Our field theory methods yield both uniqueness and stability in one. Our approach also has the advantage over Stuart's and Ball's that we are able to treat the homogeneous and cavitating solutions within the same framework.

When combined with the work of Stuart (1984) the results of Chapter 4 yield a complete description of radial cavitation without recourse to the variational methods of Ball.

Constitutive Assumptions

In the course of this thesis we will refer to a number of constitutive hypotheses on the stored energy function Φ, a list of these hypotheses is given in the Appendix.

Notation

We will write M^{3x3} for the space of all 3×3 matrices over $\mathbb{R}$. We set

$$M_+^{3x3} = \left\{ F \in M^{3x3} \; ; \; \det F > 0 \right\}$$

and denote by $SO(3)$ the special orthogonal group on $\mathbb{R}^3$.

L^p Spaces

If $E \subset \mathbb{R}^3$ is measurable, $n \geq 1$, $1 \leq p < +\infty$ then we denote by $L^p(E, \mathbb{R}^n)$ the Banach space of equivalence classes of Lebesgue measurable functions $u : E \longrightarrow \mathbb{R}^n$ with norm $\|\cdot\|_p$ defined by

$$\|u\|_p = \begin{cases} \left[\int_E |u(\underline{x})|^p \, dx \right]^{\frac{1}{p}} & 1 \leq p < +\infty \\ \operatorname*{ESSSup}_{\underline{x} \in E} |u(\underline{x})| & p = \infty \end{cases}$$

(c.f. Adams).

Sobolev Spaces

Let $E \subset \mathbb{R}^3$ be measurable and let $\underline{x} = (x_1, x_2, x_3)$. We denote by D_i the differential operator $\frac{\partial}{\partial x_i}$. If $j = (j_1, j_2, j_3)$ is a multiindex $j_i \in \mathbb{Z}$ then we write

$$D^j = D_1^{j_1} D_2^{j_2} D_3^{j_3} = \frac{\partial^{|j|}}{\partial x_1^{j_1} \partial x_2^{j_2} \partial x_3^{j_3}}$$

where $|j| = j_1 + j_2 + j_3$. If $j = (0,0,0)$ then $D^j = I$ the identity. For each integer $m \geq 1$ and $1 \leq p < +\infty$ we write $W^{m,p}(E)$ for the Sobolev space of equivalence classes of Lebesgue measurable functions u satisfying $u \in L^p(E)$ and such that all distributional derivatives of u of order less than or equal to m also lie in $L^p(E)$. $W^{m,p}(E)$ then becomes a Banach space under the norm

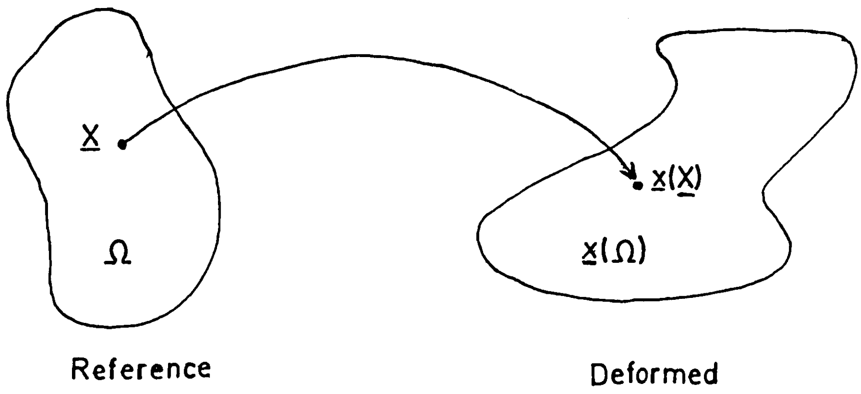
$$\|u\|_{m,p} = \left[\sum_{|j| \leq m} \|D^j u\|_p^p \right]^{\frac{1}{p}}$$

(see Adams).

CHAPTER 0

Section 1

Consider a homogeneous elastic body which in a reference configuration occupies the bounded, open, connected set $\Omega \subset \mathbb{R}^3$. In a typical deformation $\underline{x} : \Omega \longrightarrow \mathbb{R}^3$ a particle with position vector $\underline{X}$ moves to a point having position vector $\underline{x}(\underline{X})$.



If $W:M_+^{3\times3} \longrightarrow \mathbb{R}^+$ is the stored energy function of the material then the total energy E associated with the deformation $\underline{x}$ is given by

$$E(\underline{x}) = \int_\Omega W(\nabla\underline{x}(\underline{X}))\ dX \tag{0.0.1}$$

whenever

$$\det(\nabla\underline{x}(\underline{X})) > 0 \quad \text{for all } \underline{X} \in \Omega. \tag{0.0.2}$$

Condition (0.0.1) is the assumption of <u>hyperelasticity</u> (see Truesdell and Noll (1965)). We say that $\underline{x}$ is an admissible deformation if it satisfies the invertibility condition (0.0.2). Note that this is a local condition on $\underline{x}$. However, from a physical point of view we would require deformations to be globally one to one so that two distinct material points could not simultaneously occupy the same point in space. For results pertaining to the global invertibility of a deformation we refer to Meisters and Olech (1963), Ball (1981) and Weinstein (1984).

To reflect the idea that large forces are necessary to effect large extensions or compressions we require that

$$W(F) \longrightarrow \infty \quad \text{as} \quad |F| \longrightarrow \infty \quad \text{or} \quad \det F \longrightarrow 0^+ \tag{0.0.3}$$

Conditions (0.0.2) and (0.0.3) pose serious technical problems (see Antman (1984), Ball (1977), Gurtin (1981)).

We assume that W is frame indifferent i.e. that the energy of a deformation is invariant under changes in observer; this is expressed mathematically by

$$W(QF) = W(F) \quad \text{for all} \quad F \in M_+^{3x3}, \; Q \in SO(3). \tag{0.0.4}$$

We say that W is <u>isotropic</u> if in addition

$$W(FQ) = W(F) \quad \text{for all} \quad F \in M_+^{3x3}, \quad Q \in SO(3). \tag{0.0.5}$$

It can be shown that (0.0.4) and (0.0.5) hold if and only if there exists a symmetric function $\Phi: \mathbb{R}^3_{++} \longrightarrow \mathbb{R}$ satisfying

$$W(F) = \Phi(v_1, v_2, v_3) \quad \text{for all} \quad F \in M_+^{3x3}, \tag{0.0.6}$$

where

$$\mathbb{R}^3_{++} = \left\{ (c_1, c_2, c_3) \in \mathbb{R}^3 \; ; \; c_i > 0 \quad i = 1,2,3 \right\} \tag{0.0.7}$$

and where the v_i are the eigen values of $(F^TF)^{\frac{1}{2}}$, known as the <u>principle stretches</u> (for a proof see Truesdell and Noll).

We define the <u>Piola-Kirchhoff stress tensor</u> $T_R : M_+^{3x3} \longrightarrow M^{3x3}$ by

$$T_R(F) = \frac{\partial W}{\partial F}(F) \overset{\text{def}}{=} \left[\frac{\partial W(F)}{\partial F_j{}^i} \right] . \tag{0.0.8}$$

By (0.0.2), (0.0.4), (0.0.5) and (0.0.6), if W is isotropic and $F = \text{diag}(v_1, v_2, v_3)$, with $v_i > 0$ for all i , then

$$T_R(F) = \operatorname{diag}(\Phi_{,1},\Phi_{,2},\Phi_{,3}) \tag{0.0.9}$$

where $\Phi_{,i} = \Phi_{,i}(v_1,v_2,v_3)$.

The Cauchy stress tensor $T(F)$ is related to $T_R(F)$ through the formula

$$T(F) = (\det F)^{-1}\, T_R(F)F^T \tag{0.0.10}$$

The tensors T_R and T measure the force on the body per unit area in the undeformed and deformed configurations respectively.

For an elastic body with stored energy function W the equilibrium equations under zero body force are given by

$$\frac{\partial}{\partial X^\alpha}\left[\frac{\partial W}{\partial x_{,\alpha}^{i}}(\nabla \underline{x}(\underline{X}))\right] = 0 \quad \text{for } i = 1,2,3\ , \tag{0.0.11}$$

for all $\underline{X} = (X^1,X^2,X^3) \in \Omega$. These are the Euler-Lagrange equations for the functional E ((0.0.1)).

The displacement boundary value problem in elasticity consists of finding a solution $\underline{x}$ to (0.0.11) taking prescribed values on the boundary $\partial\Omega$. We now restrict attention to the case where

$$\Omega = B \overset{\text{def}}{=} \left\{\underline{X} \in \mathbb{R}^3;\ |\underline{X}| < 1\right\} \tag{0.0.12}$$

is the open unit ball and consider radial deformations; that is deformations $\underline{x}$ of the form

$$\underline{x}(\underline{X}) = \frac{r(R)}{R}\,\underline{X} \tag{0.0.13}$$

satisfying

$$\underline{x}(\underline{X}) = \lambda \underline{X} \quad \text{for } \underline{X} \in \partial\Omega, \text{ for some } \lambda \in (0,\infty), \tag{0.0.14}$$

where $R = |\underline{X}|$.

The following proposition is taken from Ball (1982) Lemma 4.1 and relates the properties of $\underline{x}$ and r as defined by (0.0.13).

Proposition 0.1

Let $1 \leq p < +\infty$ and let $\underline{x}$ be given by (0.0.13) then $\underline{x} \in W^{1,P}(B; \mathbb{R}^3)$ if and only if $r(\cdot)$ is absolutely continuous on (0,1) and

$$\int_0^1 R^2 \left[|r'(R)|^P + \left|\frac{r(R)}{R}\right|^P \right] dR < +\infty. \tag{0.1.1}$$

The weak derivatives of $\underline{x}$ are then given by

$$\nabla \underline{x}(\underline{X}) = \frac{r(R)}{R} 1 + \frac{\underline{X} \otimes \underline{X}}{R^2} \left[r'(R) - \frac{r(R)}{R} \right]. \tag{0.1.2}$$

For the proof see Ball (1982) p.566.

Following Ball (1982) we say that $\underline{x} \in W^{1,1}(B; \mathbb{R}^3)$ is a weak equilibrium solution of the displacement boundary value problem if

$\det(\nabla \underline{x}(\underline{X})) > 0$ for a.e. $\underline{X} \in B$, $\frac{\partial W}{\partial F}(\nabla \underline{x}(\cdot)) \in L^1(B; \mathbb{R}^9)$ and

$$\int_B \frac{\partial W}{\partial x^i_{,\alpha}} \Phi^i_{,\alpha}\, dX = 0 \quad \text{for all} \quad \Phi \in C_0^\infty(B; \mathbb{R}^3). \tag{0.1.3}$$

A key problem in nonlinear elasticity is to understand how assumptions on the stored energy function affect the existence and nature of weak equilibrium solutions (in general weak solutions can possess singularities even if the stored energy function is smooth (see Ball (1979))).

Ball reduces the analysis of weak equilibrium solutions of the form (0.0.13) to studying solutions of a particular ordinary differential equation by means of the following results which we state here for convenience as one theorem and whose proof is contained in Ball (1982) theorem 4.2 and proposition 6.1.

Theorem 0.2

Let $\Phi \in C^m(\mathbb{R}^3_{++})$, $m \geq 1$. Then $\underline{x}$ defined by (0.0.13) is a weak equilibrium solution if and only if $r'(R) > 0$ a.e. $R \in (0,1)$,

$$R^2\Phi_{,1}(r(R), \frac{r(R)}{R}, \frac{r(R)}{R}), R^2\Phi_{,2}(r'(R), \frac{r(R)}{R}, \frac{r(R)}{R}) \in L^1(0,1)$$

$$R^2\Phi_{,1}(r'(R), \frac{r(R)}{R}, \frac{r(R)}{R}) = 2\int_1^R \rho\Phi_{,2}(r'(\rho), \frac{r(\rho)}{\rho}, \frac{r(\rho)}{\rho})d\rho + \text{const.} \tag{0.2.1}$$

for a.e. $R \in (0,1)$. The v_i are given almost everywhere by

$$v_1 = r'(R) \ , \ v_2 = v_3 = \frac{r(R)}{R} \ . \tag{0.2.2}$$

Moreover if Φ satisfies H1 and H5 then $r \in C^m((0,1])$, $r'(R) > 0$ for every $R \in (0,1]$ and r satisfies the <u>radial equilibrium equation</u>

$$\frac{d}{dR}\left[R^2\Phi_{,1}(r'(R), \frac{r(R)}{R}, \frac{r(R)}{R})\right] = 2R\Phi_{,2}(r'(R), \frac{r(R)}{R}, \frac{r(R)}{R}) \tag{0.2.3}$$

for every $R \in (0,1]$

Notice that the <u>homogeneous deformation</u>

$$r(R) \equiv \lambda R \tag{0.2.4}$$

is always a solution of (0.2.3) and satisfies (0.0.14).

On using (0.0.6), (0.0.1) and (0.2.2) the energy corresponding to the radial deformation (0.0.13) takes the form

$$E(\underline{x}) = 4\pi I(r) = 4\pi\int_0^1 R^2\Phi(r'(R), \frac{r(R)}{R}, \frac{r(R)}{R})dR. \tag{0.2.5}$$

Notice that (0.2.3) is the Euler-Lagrange equation for (0.2.6).

To demonstrate the existence of non trivial solutions of (0.2.3) corresponding to cavitation Ball uses a variational technique, showing that the functional

I attains its infimum on a set of admissible functions A_λ where

$$A_\lambda = \left\{ r \in W^{1,1}(0,1); r(1)=\lambda, r'(R) > 0 \text{ a.e. } R \in (0,1),\ r(0) > 0 \right\}. \tag{0.2.6}$$

Our next proposition is a modified version of Ball (1982) theorem 7.1; the proof is given in the appendix.

<u>Proposition 0.3</u>

Let $\Phi \in C^m(\mathbb{R}^3_{++})$, $m \geqslant 1$, satisfy H1, H5 and E2. If r is an absolute minimiser of I on A_λ then

(i) $r'(R) > 0$ for $R \in (0,1]$, (0.3.1)

(ii) $r \in C^m((0,1])$ and satisfies (0.2.3) for every $R \in (0,1]$. (0.3.2)

Moreover if $r(0) = \lim_{R \to 0} r(R) > 0$ then

$$\lim_{R \to 0} T(r(R)) = 0 \tag{0.3.3}$$

where

$$T(r(R)) \stackrel{\text{def}}{=} \left(\frac{R}{r}\right)^2 \Phi_{,1}\left(r'(R), \frac{r(R)}{R}, \frac{r(R)}{R}\right) \tag{0.3.4}$$

is the radial component of the Cauchy stress.

From (0.0.13) we see that $r(0) > 0$ corresponds to a cavity forming at the centre of the deformed ball, and that (0.3.3) is the natural boundary condition that the cavity surface is stress free.

Ball showed that for sufficiently large values of the boundary displacement λ the minimiser r of I on A_λ satisfies $r(0) > 0$. By proposition 0.3 r is a solution of the radial equilibrium equation (0.2.3) and by theorem 0.2 $\underline{x}$ defined by (0.13) corresponds to a weak solution of the three dimensional equilibrium equations. (see appendix).

In the remainder of this thesis when referring to a <u>cavitating equilibrium solution</u> r we will mean a function $r \in C^2((0,1])$ that is a solution of

(0.2.3) on $(0,1]$ and satisfies

$$\text{(i)} \quad r'(R) > 0 \quad \text{for} \quad R \in (0,1] \quad \text{and} \tag{0.3.5}$$
$$\text{(ii)} \quad \lim_{R \to 0} T(r(R)) = 0.$$

Section 2

In this section we gather together results concerning properties of solutions $r(R)$ to the radial equilibrium equation (0.2.3). These results will be central to the arguments in the rest of this thesis.

Proposition 0.4

Let Φ satisfy H1 and let $r \in C^2((0,1])$ be a solution of (0.2.3) satisfying (0.3.5).

If $\frac{r(R_0)}{R_0} = r'(R_0) \stackrel{\text{def}}{=} \lambda_0$ for some $R_0 \in (0,1]$, $\lambda_0 \in (0,\infty)$, then $r(R) \equiv \lambda_0 R$ for $R \in (0,1]$.

Proof

Equation (0.2.3) is of the form $r'' = f(R,r,r')$ where f is C^1. Standard results for ordinary differential equations then imply that the solution $r(R)$ to the initial value problem with data $r(R_0) = \lambda_0 R_0$, $r'(R_0) = \lambda_0$ is unique. Hence $r(R) \equiv \lambda_0 R$.

Corollary 0.5

If $r \in C^2((0,1])$ with $r(R) \not\equiv \lambda R$ for any λ is a solution of (0.2.3) that satisfies (0.3.5), then $\frac{r(R)}{R}$ is a strictly monotone function on $(0,1]$ and $\frac{d}{dR}\left(\frac{r(R)}{R}\right) = \frac{1}{R}\left(r'(R) - \frac{r(R)}{R}\right)$.

In particular if $r(0) = \lim_{R \to 0} r(R) > 0$ then $r'(R) < \frac{r(R)}{R}$ for $R \in (0,1]$.

Proof

The first part of the corollary is an easy sequence of proposition 0.4. The second part then follows immediately since if $r(0) > 0$ then

$\frac{r(R)}{R} \longrightarrow \infty$ as $R \longrightarrow 0$.

We now give conditions under which the radial Cauchy stress $T(r(R))$ is monotone on any interval where $r'(R) \neq \frac{r(R)}{R}$ for any solution $r \in C^2((0,1])$ of (0.2.3).

Proposition 0.6

If Φ satisfies H1, H2 and $r \in C^2((0,1])$ is a solution of (0.2.3) which satisfies (0.3.5) then

$$\frac{dT(r(R))}{dR}\left[r'(R) - \frac{r(R)}{R}\right] \leq 0 \quad \text{for} \quad R \in (0,1] \tag{0.6.1}$$

Proof

By (0.2.3) and (0.3.4)

$$\frac{dT(r(R))}{dR} = \frac{2R^2}{r^3(R)}\left[\frac{r(R)}{R}\,\Phi_{,2}(R) - r'(R)\,\Phi_{,1}(R)\right], \tag{0.6.2}$$

where

$$\Phi_{,i}(R) = \Phi_{,i}(r'(R), \frac{r(R)}{R}, \frac{r(R)}{R}). \tag{0.6.3}$$

The result then follows by H2 and (0.2.2).

Related to the above we define the inverse Cauchy stress $\tilde{T}(r(R))$ by

$$\tilde{T}(r(R)) = \Phi(r'(R), \frac{r(R)}{R}, \frac{r(R)}{R}) - r'(R)\Phi_{,1}(r'(R), \frac{r(R)}{R}, \frac{r(R)}{R}). \tag{0.6.4}$$

We refer to Ball (1982) for an interpretation of $\tilde{T}$ and the proof of the following analogue of proposition 0.6.

Proposition 0.7

If Φ satisfies H1, H2 and $r \in C^2((0,1])$ is a solution of (0.2.3) satisfying (0.3.5), then

$$\frac{d\tilde{T}(r(R))}{dR}\left(r'(R) - \frac{r(R)}{R}\right) \geq 0 \quad \text{for} \quad R \in (0,1]. \tag{0.7.1}$$

A third related lyapunov function is given by the following identity

$$\frac{d}{dR}\left\{R^3\left[\Phi\left(r'(R),\frac{r(R)}{R},\frac{r(R)}{R}\right) + \left(\frac{r(R)}{R} - r'(R)\right)\Phi_{,1}\left(r'(R),\frac{r(R)}{R},\frac{r(R)}{R}\right)\right]\right\}$$
$$= 3R^2\Phi\left(r'(R),\frac{r(R)}{R},\frac{r(R)}{R}\right). \tag{0.7.2}$$

For future reference we introduce the notation

$$H(X,Y) \overset{\text{def}}{=} \Phi(X,Y,Y) + (Y-X)\Phi_{,1}(X,Y,Y). \tag{0.7.3}$$

It is noted in Ball (1982) that (0.7.2) is the radial version of the following three dimensional conservation law

$$\frac{\partial}{\partial X^\alpha}\left[X^\alpha W - \frac{\partial W}{\partial x^j_{,\alpha}}\left(X^\beta x^j_{,\beta} - x^j\right)\right] = 3W, \tag{0.7.4}$$

(see Green (1973)). Equation (o.7.4) was recently used by Knops and Stuart (1984) to prove the uniqueness of smooth equilibrium solutions to the displacement boundary value problem of elasticity for star-shaped domains under assumptions of quasiconvexity.

Proposition 0.8

If Φ satisfies H1, H7 and $r \in C^2((0,1])$ is a solution of (0.2.3) satisfying $r'(R) < \frac{r(R)}{R}$ (respectively $r'(R) > \frac{r(R)}{R}$) for $R \in (0,1]$ then $r''(R) > 0$ (respectively $r''(R) < 0$) for $R \in (0,1]$.

Proof

The proof is an immediate consequence of H7, (0.2.3) and corollary 0.5.

Proposition 0.9

Let Φ satisfy H1, H2 and H5. If $r \in C^2((0,1])$ is a cavitating equilibrium solution then r is extendable to $r \in C^2((0,\infty))$ as a solution of (0.2.3) and satisfies

$$\text{(a)} \quad \frac{r(R)}{R} > r'(R) > 0 \quad \text{for} \quad R \in (0,\infty), \tag{0.9.1}$$

$$\text{(b)} \quad \lim_{R \to 0} \frac{r(R)}{R} = \lim_{R \to \infty} r'(R) = \lambda_c \quad \text{for some } \lambda_c \in [1,\infty). \tag{0.9.2}$$

Proof

By the continuation principle (see e.g. Hirsch and Smale) r may be extended to a maximal interval of existence $(0,\partial)$, $\partial > 1$, as a solution of (0.2.3) satisfying (0.7.1). We suppose for a contradiction that is finite; then one of the following cases must occur

$$\text{(i)} \quad \lim_{R \to \partial} \frac{r(R)}{R} = \infty,$$

$$\text{(ii)} \quad \lim_{R \to \partial} \frac{r(R)}{R} = 0,$$

$$\text{(iii)} \quad \lim_{R \to \partial} r'(R) = \infty,$$

$$\text{(iv)} \quad \lim_{R \to \partial} r'(R) = 0.$$

It follows from corollary 0.5 that (i) cannot occur; the same is true for (iii) as clearly (iii) implies (i) (also by corollary 5).

If (ii) holds then there exists $R_0 \in (0,\infty)$ satisfying $\frac{r(R_0)}{R_0} = 1$ since $\lim_{R \to 0} \frac{r(R)}{R} = \infty$. On applying proposition 0.6 with (0.9.1) we conclude that $T(r(R))$ is non-decreasing and hence

$$0 = T(r(0)) = \lim_{R \to 0} T(r(R)) \leqslant T(r(R_0)) = \Phi_{,1}(r'(R_0),1,1) \leqslant \Phi_{,1}(1,1,1) = 0 \tag{0.9.3}$$

a contradiction (equality holds in the last term in (0.9.3) since the reference configuration is a natural state). Now suppose that (iv) holds; it then follows that $\frac{r(R)}{R} \searrow b > 0$ as $R \longrightarrow \partial$ as (ii) is false. Assumption H5 gives the existence of $a \in (0,\infty)$ satisfying

$$\Phi_{,1}(a,b,b) < 0. \tag{0.9.4}$$

Then H1 implies that for R sufficiently close to ∂

$$T(r(R)) = \left(\frac{R}{r(R)}\right)^2 \Phi_{,1}\left(r'(R),\frac{r(R)}{R},\frac{r(R)}{R}\right) < \left(\frac{R}{r(R)}\right)^2 \Phi_{,1}\left(a,\frac{r(R)}{R},\frac{r(R)}{R}\right). \tag{0.9.5}$$

As $\frac{r(R)}{R} \longrightarrow b$ as $R \longrightarrow \partial$, (0.9.4) implies that $T(r(R))$ is negative for R sufficiently close to ∂, which contradicts proposition 0.6 as $T(r(0)) = 0$. Hence (iv) cannot hold and $\partial = \infty$.

We next prove part (b) of the proposition.

By (0.9.1) and corollary 0.5 $\frac{r(R)}{R}$ is monotone decreasing, and so

$$\frac{r(R)}{R} \searrow \lambda_c \quad \text{as} \quad R \longrightarrow \infty \quad \text{for some} \quad \lambda_c \in [0,\infty). \tag{0.9.6}$$

An analogous argument to that used in the negation of case (ii) then implies that $\lambda_c \in [1,\infty)$. Finally, the monotonicity of $T(r(R))$ together with

$$T(r(R)) < \left(\frac{R}{r}\right)^2 \Phi_{,1}\left(\frac{r}{R}, \frac{r}{R}, \frac{r}{R}\right) < \text{const.}$$

(which is a consequence of H1 and (0.9.6)) implies that $\lim_{R \to \infty} T(r(R)) = d$ for some $d \in [0,\infty)$. We suppose for a contradiction that $\lim_{R \to \infty} r'(R) \neq \lambda_c$ then there exist a sequence $R_n \longrightarrow \infty$ as $n \longrightarrow \infty$ and $\varepsilon_0 \in (0,\infty)$ such that $|r'(R_n) - \lambda_c| \geq \varepsilon_0$ for all n. We assume without loss of generality that $r'(R_n) \geq \lambda_c + \varepsilon_0$ for all n (an exactly analogous argument holds in the case $r'(R_n) \leq \lambda_c - \varepsilon_0$ for all n). It then follows from H1 that

$$T(r(R_n)) \geq \left(\frac{R_n}{r(R_n)}\right)^2 \Phi_{,1}\left(\lambda_c + \varepsilon_o, \frac{r(R_n)}{R_n}, \frac{r(R_n)}{R_n}\right) \tag{0.9.7}$$

for all n. Finally on passing to the limit in (0.9.7) we obtain again by H1 that

$$d \geq \frac{1}{\lambda_c^2} \Phi_{,1}(\lambda_c + \varepsilon_o, \lambda_c, \lambda_c) > d,$$

which is a contradiction.

Remark 0.10

If H2$^+$ holds then $\lambda_c \in (1,\infty)$ because $T(r(R))$ is then strictly monotone increasing.

Corollary 0.11

The results of proposition 0.9 hold if H5 is replaced by H7.

Proof

The proof follows from proposition 0.8, the continuation principle and analogous arguments to those used in proposition 0.9 on noting that

$$0 < r'(R) < r'(s) < \frac{r(s)}{s} < \frac{r(R)}{R} \quad \text{for} \quad s < R.$$

(see also Ball (1982) p.601).

Proposition 0.12

Let Φ satisfy H1, H2 and H3. If $r \in C^2((0,1])$ is a solution of (0.2.3) satisfying (0.3.1) with $r'(R) < \frac{r(R)}{R}$ for $R \in (0,1]$, then there exists $M > 0$ such that

$$0 < |r'(R)| = r'(R) \leq M \quad \text{for} \quad R \in (0,1]. \tag{0.12.1}$$

Proof

We assume without loss of generality that Φ satisfies the first condition of H3; otherwise exactly analogous arguments hold on using the inverse Cauchy stress $\tilde{T}$ and proposition 0.7 instead of the radial Cauchy stress T. It follows from proposition 0.6 that $T(r(R))$ is non decreasing.

Let $\alpha = T(r(1))$. We assume for a contradiction that (0.12.1) does not hold for any M. This implies the existence of a sequence $\{R_n\} \in (0,1]$, $R_n \longrightarrow 0$ as $n \longrightarrow \infty$, satisfying $n < r'(R_n) < \frac{r(R_n)}{R_n}$ for all n. It then follows from H3 that $T(r(R_n)) \longrightarrow \infty$ as $n \longrightarrow \infty$ and so $T(r(R_N)) > \alpha$ for some N, contradicting the fact that T is non decreasing.

Remark 0.13

If Φ satisfies H1 and H7 then the above result follows trivially from proposition 0.8.

The following fundamental theorem embodies some of the central ideas associated with the phenomenon of cavitation.

Theorem 0.14

Suppose that Φ satisfies E1, E2, H1, H2$^+$, H3, H4, H5 and that for some $\lambda > 0$ there exists a cavitating equilibrium solution $r_c \in C^2((0,1])$.
Then

(i) r_c is unique and extendable to $r_c \in C^2((0,\infty))$ as a solution of (0.2.3),

(ii) $\lim\limits_{R \to \infty} \frac{r_c(R)}{R} = \lambda_c$ for some $\lambda_c \in (1,\infty)$,

(iii) if $\lambda \leqslant \lambda_c$ then $r(R) \equiv \lambda R$ is the unique global minimiser of I on A_λ,

(iv) if $\mu > \lambda_c$ then the global minimiser r_μ of I on A_λ exists and satisfies $r_\mu(0) > 0$. Moreover

$$r_\mu(R) \equiv \partial r_c(\tfrac{R}{\partial}) \quad \text{for} \quad R \in (0,1],$$

where ∂ is the unique root of $\partial r_c(\frac{1}{\partial}) = \mu$.

($\lambda_c = \lambda_{crit}$ as defined in Ball (1982) p.601).

The proof of this theorem is given at the end of Chapter 3. The theorem also holds with H2 in place of $H2^+$ with the exception that in this case $\lambda_c \in [1,\infty)$. (see remark 0.10).

Corollary 0.15

The result holds with H3 and H4 replaced by H7.

We refer to Chapter 1 for results concerning the existence of cavitating equilibria. The next proposition uses the conservation law (0.7.2) and will play a central role in our analysis.

Proposition 0.16

Suppose Φ satisfies H1, H2 and H3 and that $r_c \in C^2((0,1])$ is a cavitating equilibrium solution. Then

$$\text{(i)} \quad \lim_{R \to 0} R^3 \left[\Phi(r_c', \tfrac{r_c}{R}, \tfrac{r_c}{R}) + (\tfrac{r_c}{R} - r_c')\Phi_{,1}(r_c', \tfrac{r_c}{R}, \tfrac{r_c}{R})\right] = 0 \qquad (0.16.1)$$

$$\text{(ii)} \quad I(r_c) = \tfrac{1}{3}\left[\Phi(r_c'(1), r_c(1), r_c(1)) + (r_c(1) - r_c'(1))\Phi_{,1}(r_c'(1), r_c(1), r_c(1))\right] \qquad (0.16.2)$$

In particular any cavitating equilibrium solution has finite energy.

Proof

Equation (0.7.2) implies that for $\tau \in (0,1)$

$$\tau^3\Phi(\tau) + 3\int_\tau^1 R^2\Phi(R)dR = \Phi(1) + (r_c(1) - r_c'(1))\Phi_{,1}(1) + \tau^3(r_c'(\tau) - \tfrac{r_c(\tau)}{\tau})\Phi_{,1}(\tau). \qquad (0.16.3)$$

The last term on the right hand side of (0.16.3) may be written as

$$\left[\tau r_c'(\tau) - r_c(\tau)\right] r_c^2(\tau) \left[\frac{\tau}{r_c(\tau)}\right]^2 \Phi_{,1}(\tau) \qquad (0.16.4)$$

and it follows from (0.3.3) that the limit as $\tau \longrightarrow 0$ of (0.16.4) is zero as $r_c'(R)$ is bounded by proposition 0.12. Hence the limit as $\tau \longrightarrow 0$ of the right hand side of (0.16.3) exists. But the left hand side of (0.16.3) is the sum of two positive terms; so by the monotone convergence theorem

$$R^2\Phi(r_c', \frac{r_c}{R}, \frac{r_c}{R}) \in L^1(0,1).$$

Therefore $\lim_{\tau \to 0} \tau^3\Phi(\tau)$ exists and is equal to zero.

Remark 0.17

Proposition 0.16 holds with H2 and H3 replaced by H7 (this follows on using remark 0.13 in place of proposition 0.12 in the above arguements).

CHAPTER 1

In this chapter we prove the existence of energy minimisers for the displacement boundary value and traction problems and the existence of cavitating minimisers for appropriate boundary data: the homogeneous and inhomogeneous cases are considered in sections 1 and 2 respectively. The chapter is concluded with the determination of the critical load for an incompressible inhomogeneous material.

1. Homogeneous Case

Our first proposition concerns the existence of energy minimisers for the displacement boundary value problem.

Proposition 1.1

Let Φ satisfy E1 and H1 and let I be defined by

$$I(r) = \int_0^1 R^2\Phi(r'(R), \frac{r(R)}{R}, \frac{r(R)}{R})\ dR. \tag{1.1.1}$$

Then I attains its infimum on A_λ (where A_λ is defined by (0.2.6)).

Proof

Let $\{y_n\}$ be a minimising sequence for I on A_λ and let $\beta = \operatorname*{Inf}_{A_\lambda} I$.

Assumption H1 implies that for each positive integer m

$$\int_{\frac{1}{m}}^{1} \left(\frac{1}{m}\right)^2 \cdot \psi(y_n')dR \leq I(y_n) \leq \text{constant} \quad \text{for all n.} \tag{1.1.2}$$

Using the De la Vallee Poussin criterion (see Cesari p.329) we choose the following sequences inductively;

$$\{y_{m;n}\}_{n=1}^{\infty} \text{ to be a subsequence of } \{y_{m-1,n}\}_{n=1}^{\infty} \text{ satisfying} \tag{1.1.3}$$

$$y'_{m,n} \xrightarrow{L^1(\frac{1}{m},1)} z_m \text{ as } n \longrightarrow \infty \tag{1.1.4}$$

for some $Z_m \in L^1(\frac{1}{m},1)$ and we define $\{y_{1,n}\}$ by $y_{1,n} = y_n$ for all n.
We define the function Z by

$$Z(R) = Z_k(R) \text{ where k is chosen so that } R \in (\tfrac{1}{k},1). \tag{1.1.5}$$

The function Z is then well defined for a.e. R since if $m_1 > m_2$ then

$$y'_{m_i,n} \xrightarrow{L^1(\frac{1}{m_i},1)} Z_{m_i} \quad \text{as } n \longrightarrow \infty, \; i = 1,2, \tag{1.1.6}$$

so by the uniqueness of weak limits $Z_{m_1}(R) = Z_{m_2}(R)$ for a.e. $R \in (\frac{1}{m_1},1)$.
We now set

$$y(R) = \lambda - \int_R^1 z(s)\, ds \tag{1.1.7}$$

and

$$r_m = y_{m,m} \quad \text{for all m.} \tag{1.1.8}$$

The sequence $\{r_m\}$ defined by (1.1.8) then satisfies

$$r_m \xrightarrow{W^{1,1}(\delta,1)} y \quad \text{as } m \longrightarrow \infty \tag{1.1.9}$$

for each $\delta \in (0,1)$.

We extend the definition of Φ by setting $\Phi(v_1,v_2,v_3) = \infty$ if $v_i \leq 0$ for any i so that for each $R \in (0,1)$ $g(R,\cdot,\cdot)$ defined by

$$g(R,r,r') = R^2\Phi(r', \frac{r}{R}, \frac{r}{R})$$

becomes a continuous function from $\mathbb{R} \times \mathbb{R} \longrightarrow \overline{\mathbb{R}}$. Then by E1, H1 and using a standard lower semicontinuity theorem (c.f. Ball, Currie and Olver (1980) theorem 5.4) we conclude that

$$\int_\delta^1 R^2\Phi(y'(R), \frac{y(R)}{R}, \frac{y(R)}{R})dR \leq \varliminf_{m\to\infty} \int_\delta^1 R^2\Phi(r'_m(R), \frac{r_m(R)}{R}, \frac{r_m(R)}{R})dR \tag{1.1.10}$$

for each $\delta \in (0,1)$. Since the $\{r_m\}$ are a subsequence of a minimising sequence for I on A_λ

$$\int_\delta^1 R^2\Phi(y'(R), \frac{y(R)}{R}, \frac{y(R)}{R})dR \leq \beta = \operatorname*{Inf}_{A_\lambda} I \quad \text{for each } \delta \in (0,1). \tag{1.1.11}$$

Using the fact that Φ is positive we obtain the monotone convergence theorem that

$$\int_0^1 R^2\Phi(y'(R), \frac{y(R)}{R}, \frac{y(R)}{R})\, dR \leq \beta. \tag{1.1.12}$$

to complete the proof we show that $y \in A_\lambda$ so that equality holds in (1.1.12). It follows from E1, (1.1.5) and (1.1.6) that $y'(R) > 0$ for a.e. $R \in (0,1)$. Clearly $y(1) = \lambda$ and as

$$\int_\partial^1 |y'|\, ds = \int_\partial^1 y' \quad ds \leq 2\lambda \quad \text{for each } \partial \in (0,1)$$

the monotone convergence theorem implies that $y' \in L^1(0,1)$ and hence $y \in W^{1,1}(0,1)$. Finally (1.1.9) implies that

$$r_m \xrightarrow{C([\partial,1])} y \quad \text{as } m \longrightarrow \infty \text{ for each } \partial \in (0,1);$$

hence $y(R) \geq 0$ for $R \in (0,1)$ and so $y(0) \geq 0$. This establishes that $y \in A_\lambda$.

We next show the existence of energy minimising deformations for the dead load traction problem, this corresponds to the boundary condition

$$T_R(\nabla \underline{x}\ (\underline{X}))\underline{n}(\underline{X}) = P\underline{X} \text{ for } \underline{X} \in \partial B, \tag{1.1.13}$$

where T_R is the Piola-Kirchhoff stress tensor, $\underline{n}(\underline{X})$ is the unit normal to B at the point $\underline{X}$ and P is a given constant.

The associated energy functional for radial deformations is then given by

$$I^P(r) = \int_0^1 R^2\Phi(r'(R), \frac{r(R)}{R}, \frac{r(R)}{R})\, dR - Pr(1). \tag{1.1.14}$$

Proposition 1.2

Let Φ satisfy E1, H1 and let $\mathcal{B}$ be defined by

$$\mathcal{B} = \left\{ r \in W^{1,1}(0,1) \ ; \ r(0) \geq 0 \ , \ r'(R) > 0 \right\}. \tag{1.2.1}$$

Then I^P attains its infimum on $\mathcal{B}$.

Proof

For each $r \in \mathcal{B}$

$$I^P(r) \geq K_1 \int_0^1 R^2 \frac{r(R)}{R}\, dR + K_2 \int_0^1 R^2 r'(R)dR + K_3 - Pr(1) + \frac{1}{2}\int_0^1 R^2\psi(r'(R))dR \tag{1.2.2}$$

by E1, where K_1 and K_2 may be chosen to be arbitrarily large by suitably altering K_3. Integrating the second term on the right hand side of (1.2.2) by parts gives

$$I^P(r) \geq \frac{1}{2}\int_0^1 R^2\psi(r')dR + K_1 \int_0^1 RrdR + K_2 r(1) - 2K_2 \int_0^1 RrdR + K_3 - Pr(1). \tag{1.2.3}$$

Choosing $K_2 > P$ and $K_1 > 2K_2$ we obtain

$$I^P(r) \geq \frac{1}{2}\int_0^1 R^2\psi(r')dR + (K_2 - P)r(1) + K_3. \tag{1.2.4}$$

The result now follows using exactly analogous arguments to proposition 1.1 on noting that $r_n(1)$ is bounded by (1.2.4) for any minimising sequence $\{r_n\}$.

With a view to proving the existence of cavitating minimisers we will establish conditions under which any solution $r \in C^2((0,1])$ of (0.2.3) satisfying (0.3.1), $r(1) = \lambda$ and $r(0) = \lim\limits_{R \to 0} r(R) = 0$ must be identically equal to λR. We first state a preparatory result, the proof of which is contained in Ball (1982) theorem 6.5.

Proposition 1.3

Let Φ satisfy H1-H4 and let r $C^2((0,1])$ be a solution of (0.2.3) satisfying (0.3.5) with $\lim\limits_{R \to 0} r(R) = r(0) = 0$. Then

$$r \in C^1([0,1]) \cap C^2((0,1]) \tag{1.3.1}$$

and

$$r'(0) = \lim_{R \to 0} r'(R) = \lim_{R \to 0} \frac{r(R)}{R} = l \tag{1.3.2}$$

for some $l \in (0,\infty)$.

Proposition 1.4

Let Φ satisfy H1-H4 and let $r \in C^2((0,1])$, $r(R) \not\equiv \lambda R$ be a solution of (0.2.3) satisfying (0.3.5) with $\lim_{R \to 0} r(R) = r(0) = 0$, $r(1) = \lambda$.
Then

$$I(r) < I(\lambda R). \tag{1.4.1}$$

Proof

The proof follows analogous lines to that of proposition 0.10.
It follows from (0.7.2) that for any $\tau \in (0,1)$

$$\tau^3\Phi(\tau) + 3\int_\tau^1 R^2\Phi(R)dR = \Phi(1) + (\lambda - r'(1))\Phi_{,1}(1) + \tau^3(r'(\tau) - \frac{r(\tau)}{\tau})\,\Phi_{,1}(\tau). \tag{1.4.2}$$

The last term on the right hand side of (1.4.2) may be written as

$$r^2(\tau)(\tau r'(\tau) - r(\tau))\left(\frac{\tau^2}{r^2(\tau)}\,\Phi_{,1}(\tau)\right) \tag{1.4.3}$$

and on using proposition 1.3 and the fact that $r(0) = 0$, we conclude that (1.4.3) tends to zero as $\tau \longrightarrow 0$. Since the left hand side of (1.4.2) is the sum of two positive terms one of which is monotone, the limit as $\tau \longrightarrow 0$ of each of them exists. By the monotone convergence theorem $I(r) < +\infty$ and so $\lim_{\tau \to 0} \tau^3\Phi(\tau) = 0$. Equation (1.4.2) then takes the form

$$I(r) = \frac{1}{3}\left[\Phi(r'(1),\lambda,\lambda) + (\lambda - r'(1))\Phi_{,1}(r'(1),\lambda,\lambda)\right] \tag{1.4.4}$$

and on using H1 we obtain

$$I(r) < \frac{\Phi(\lambda,\lambda,\lambda)}{3} = \int_0^1 R^2\Phi(\lambda,\lambda,\lambda)dR = I(\lambda R) \tag{1.4.5}$$

as required $(r'(1) \neq \lambda$ by proposition 0.4).

Proposition 1.5

Let Φ satisfy H1-H4 and let $r \in C^2((0,1])$, $r(R) \not\equiv \lambda R$ be a solution of (0.2.3) satisfying (0.3.5) with $r(1) = \lambda$ and $\lim_{R \to 0} r(R) = r(0) = 0$. Then

$$I(\lambda R) < I(r). \tag{1.5.1}$$

Proof

It follows from H1 and proposition 0.4 that

$$R^2\Phi(r', \frac{r}{R}, \frac{r}{R}) > R^2\Big(\Phi(\frac{r}{R}, \frac{r}{R}, \frac{r}{R}) + (r' - \frac{r}{R})\,\Phi_{,1}(\frac{r}{R}, \frac{r}{R}, \frac{r}{R})\Big) \tag{1.5.2}$$

for $R \in (0,1]$. Then for $\tau \in (0,1)$

$$\int_\tau^1 R^2\Phi(r', \frac{r}{R}, \frac{r}{R})dR = \left[\frac{R^3}{3}\,\Phi(\frac{r}{R}, \frac{r}{R}, \frac{r}{R})\right]_\tau^1 \tag{1.5.3}$$

and letting $\tau \longrightarrow 0$ by proposition 1.3 part (1.3.2) we obtain

$$I(r) = \int_0^1 R^2\Phi(r', \frac{r}{R}, \frac{r}{R})dR > \frac{1}{3}\,\Phi(\lambda,\lambda,\lambda) = I(\lambda R), \tag{1.5.4}$$

as required.

The observation that H1 implies (1.5.3) was made by Ball.
On combining the last two propositions we obtain the following result.

Proposition 1.6

Let Φ satisfy H1-H4 and let $r \in C^2((0,1])$ be a solution of (0.2.3) satisfying (0.3.1) with $r(1) = \lambda$ and $\lim_{R \to 0} r(R) = r(0) = 0$. Then

$r(R) \equiv \lambda R$.

Proof

We suppose for a contradiction that $r(R) \not\equiv \lambda R$; then applying propositions 1.5 and 1.4 we obtain $I(r) > I(\lambda R)$ and $I(r) < I(\lambda R)$.

Remark 1.7

We refer to Ball (1982) for an alternative proof of proposition 1.6 and the case when H3 and H4 are replaced by H7.

Proposition 1.6 is in the spirit of a recent result by Knops and Stuart (1984) concerning the uniqueness of smooth solutions to the equilibrium equations of elasticity.

Our next result concerns the existence of cavitating minimisers for the displacement boundary value problem.

Proposition 1.8

Let Φ satisfy H1-H4, H9, H10, E1 and E2. Then any minimiser r of I on A_λ satisfies $r(0) > 0$ for λ sufficiently large.

Proof

A minimiser r exists by proposition 1.1 and is a smooth solution of the radial equilibrium equation by proposition 0.3. It follows from proposition 1.6 that if $r(0) = 0$ then $r(R) \equiv \lambda R$. To prove the proposition it therefore suffices to exhibit a function $\tilde{r} \in A_\lambda$ satisfying $\tilde{r}(0) > 0$ and having less energy than the homogeneous deformation for sufficiently large λ. To this end we choose the following test function

$$\tilde{r}(R) = \begin{cases} \left[R^3 + \varepsilon^3\right]^{\frac{1}{3}} & \text{if } R \in [0,\partial] \quad (1.8.1) \\ \lambda R & \text{if } R \in (\partial,1] \quad (1.8.2) \end{cases}$$

where $\partial = \dfrac{\varepsilon}{(\lambda^3 - 1)^{\frac{1}{3}}}$. It is easily checked that $\tilde{r} \in A_\lambda$. The difference in energies is then given by

$$\Delta E = I(\tilde{r}) - I(\lambda R) = \int_0^\partial R^2 \left[\Phi(\tilde{r}', \frac{\tilde{r}}{R}, \frac{\tilde{r}}{R}) - \Phi(\lambda,\lambda,\lambda)\right] dR. \tag{1.8.3}$$

Setting $v = \dfrac{\tilde{r}}{R}$ and using the definition of ∂, (1.8.3) takes the form

$$\Delta E = \varepsilon^3 \int_\lambda^\infty \frac{v^2}{(v^3 - 1)^2} \Phi(v)dv - \frac{\varepsilon^3 \Phi(\lambda,\lambda,\lambda)}{3(\lambda^3 - 1)} \tag{1.8.4}$$

$$\varepsilon^3 \left[\int_\lambda^\infty \frac{v^2}{(v \; -1)^2} \Phi(v)dv - \frac{\Phi(\lambda,\lambda,\lambda)}{3\lambda^3}\right] \tag{1.8.5}$$

Hence by H9 and H10 ΔE is negative for λ sufficiently large, as the first term in (1.8.5) is monotone decreasing and the second bounded away from zero.

Remark 1.9

The above proposition holds with H2-H4 replaced by H7 or by any conditions under which proposition 1.6 holds (for a variety of such results see Ball (1982) chapter 6).

We next prove the analogue of proposition 1.8 for the traction boundary value problem.

Proposition 1.10

Let Φ satisfy H1-H4, H9, H10, E1 and E2. Then any minimiser r of I^P on satisfies $r(0) > 0$ for P sufficiently large.

Proof

The argument follows the proof of proposition 1.8 with the exception that we use two sets of test functions, the first given by

$$\tilde{r}_\mu(R) = \left[R^3 + \mu^3\right]^{\frac{1}{3}}. \tag{1.10.1}$$

Then the difference in energies between $\tilde{r}_\mu(R)$ and μR is given by

$$\Delta E = I^P(\tilde{r}_\mu) - I^P(\mu R) = \mu^3 \int_{\hat{\mu}}^{\infty} \frac{v^2}{(v^3-1)^2} \hat{\Phi}(v)dv - \Phi(\mu,\mu,\mu) - P(\hat{\mu}-\mu), \tag{1.10.2}$$

where

$$\hat{\mu} = \tilde{r}_\mu(1) = (1 + \mu^3)^{\frac{1}{3}} \tag{1.10.3}$$

and we have again set $v = \frac{\tilde{r}_\mu}{R}$. So if P is positive

$$\Delta E < \mu^3 \left[\int_{\hat{\mu}}^{\infty} \frac{v^2}{(v^3-1)^2} \hat{\Phi}(v)dv - \frac{\Phi(\mu,\mu,\mu)}{\mu^3} \right] \tag{1.10.4}$$

and so ΔE is negative by H9 and H10 provided $\mu > k$ for some constant $k \in (0,\infty)$. For $\mu < k$ we use the following test functions

$$\hat{r}_\mu(R) = \left[R^3+\partial^3\right]^{\frac{1}{3}} \quad \text{where} \quad \partial^3 = (2+\mu)^3-1. \tag{1.10.5}$$

The energy difference then takes the form

$$\Delta\hat{E} = I^P(\hat{r}_\mu) - I^P(\mu R) = \partial^3 \int_{(2+\mu)}^{\infty} \frac{v^2}{(v^3-1)^2}\,\hat{\Phi}(v)dv - \Phi(\mu,\mu,\mu) - 2P \tag{1.10.6}$$

The integral term in (1.10.6) is clearly bounded for $\mu \leq k$ by H9 and (1.10.5) and so the right hand side of (1.10.6) is negative for P sufficiently large. Thus μR cannot be the minimiser of I^P for any μ and the proposition is proved.

2. Inhomogeneous Case

Existence of Minimisers

For a radially inhomogeneous material the stored energy function has explicit dependence on R. We can demonstrate the existence of energy minimisers for the displacement and traction boundary value problems provided we assume that Φ satisfies

$$\Phi(R,v_1,v_2,v_3) \geq C(R) \sum_{i=1}^{3} \Psi(v_i),$$

where $C \in C^1([0,1])$ is strictly positive and satisfies $\left|C'(R)\right| \leq M \frac{C(R)}{R}$ for some constant M and where Ψ satisfies the conditions of E1 (the proof follows similar lines to propositions 1.1 and 1.2).

Existence of Cavitating Minimisers

The techniques of section 1 are not directly applicable in the inhomogeneous case. However they do become applicable if we assume for instance that Φ satisfies

$$k_2\tilde{\Phi}(v_1,v_2,v_3) \leq \Phi(R,v_1,v_2,v_3) \leq k_1\tilde{\Phi}(v_1,v_2,v_3)$$

for some $k_1, k_2 > 0$, where $\tilde{\Phi}$ is a homogeneous stored energy function satisfying the conditions of section 1. Under this assumption we can then show the existence of cavitating minimisers to the displacement and traction boundary value problems for sufficiently large boundary data.

In the final section of this chapter we calculate the critical load for an incompressible inhomogeneous material. However we first introduce some of the relevant ideas of chapter 0 section 1 for incompressible stored energy functions.

3. Incompressible Elasticity

In incompressible elasticity any admissible deformation $\underline{x}(\underline{X})$ must satisfy the pointwise constraint

$$\det(\nabla\underline{x}(\underline{X})) = 1 \tag{1.11.1}$$

Hence an inhomogeneous incompressible stored energy function $W(\underline{X},F)$ corresponds to a map

$$W(\underline{X},\cdot) : M_1^{3x3} \longrightarrow \mathbb{R} \quad \text{for each } \underline{X}\in\Omega, \tag{1.11.2}$$

where

$$M_1^{3x3} = \left\{ F\in M^{3x3} \;;\; \det F = 1 \right\}. \tag{1.11.3}$$

Frame indifference and isotropy are defined in an exactly analogous manner to the compressible case with the exception that (0.0.4) and (0.0.5) are only required to hold for $F\in M_1^{3x3}$ and for each $\underline{X}\in\Omega$.
Any such stored energy function W may be extended to the whole of M_+^{3x3}, for example by setting

$$\hat{W}(\underline{X},F) = W(\underline{X},(\det F)^{-\frac{1}{3}} F),$$

as noted in Ball (1982).

For an incompressible material the Piola-Kirchhoff and Cauchy stress tensors are defined by

$$T_R(\underline{X},F) = -PF^{-T} + T_R^{\star}(\underline{X},F) \tag{1.11.4}$$

and

$$T(\underline{X},F) = -P1 + T^{\star}(\underline{X},F) \tag{1.11.5}$$

for $F \in M_1^{3x3}$ and $\underline{X} \in \Omega$, where p is an arbitrary hydrostatic pressure and

$$T_R^{\ *}(\underline{X},F) \overset{\text{def}}{=} \frac{\partial}{\partial F} W(\underline{X},F)$$

and

$$T_R^{*}(\underline{X},F) \overset{\text{def}}{=} \frac{\partial}{\partial F} W(\underline{X},F)F^T.$$

are the Piola-Kirchhoff and Cauchy <u>extra</u> <u>stress</u> tensors respectively (in order that (1.11.4) and (1.11.5) be well defined it is necessary for W to be extended to M_+^{3x3}).

Analogously to the compressible homogeneous case $W(\underline{X},F)$ is isotropic if, and only if, there exists a symmetric function

$$\Phi(\underline{X},\cdot,\cdot,\cdot)\colon D^3 \longrightarrow \mathbb{R} \quad \text{for each } \underline{X} \in \Omega,$$

satisfying

$$\Phi(\underline{X},v_1,v_2,v_3) = W(\underline{X},F) \quad \text{for each } \underline{X} \in \Omega,\ F \in M_1^{3x3}, \tag{1.11.6}$$

where

$$D^3 = \left\{(c_1,c_2,c_3) \in R^3 \ ; \ c_1,c_2,c_3 = 1\right\}$$

and the v_i are the eigenvalues of $(F^TF)^{\frac{1}{2}}$.

We now restrict attention to the case of radial deformations of a ball of incompressible, inhomogeneous material and to the particular case in which the inhomogeneity is radial. The constraint (1.11.1) implies that any radial deformation of the form (0.0.13) must satisfy

$$r'(R)\left[\frac{r(R)}{R}\right]^2 = 1 \quad \text{for a.e. } R \in (0,1)$$

and hence the only admissible deformations are given by

$$r(R) = (R^3 + A^3)^{\frac{1}{3}} \tag{1.11.7}$$

where $A \geq 0$ is a constant.

For the dead load traction problem in which the prescribed traction is radial of constant magnitude P

$$T_R(\underline{X},\nabla \underline{x}(X)) = P1 \quad \text{for } \underline{X} \in \partial B. \tag{1.11.8}$$

The corresponding radial Cauchy stress is then given by

$$T(R) = \frac{P}{(1+A^3)^{2/3}} + \int_1^R \frac{2s^2}{r^3}\left[\frac{r}{s}\,\Phi_{,2}\left(s,\left(\frac{s}{r}\right)^2,\frac{r}{s},\frac{r}{s}\right) - \left(\frac{s}{r}\right)^2\Phi_{,1}\left(s,\left(\frac{s}{r}\right)^2,\frac{r}{s},\frac{r}{s}\right)\right] ds$$

(see Ball (1982)), where r is given by (1.11.7) and

$$\Phi_{,i}(R,v_1,v_2,v_3) \text{ denotes } \frac{\partial \Phi}{\partial v_i} \quad i = 1,2,3.$$

We require that the cavity surface be stress free; hence $T(0) = 0$ and A is a root of

$$P = (1+A^3)^{\frac{2}{3}} \int_0^1 \frac{2s^2}{r^3}\left[\frac{r}{s}\,\Phi_{,2}(s) - \left(\frac{s}{r}\right)^2\Phi_{,1}(s)\right] ds.$$

It then follows that

$$P = (1+A^3)^{\frac{2}{3}} \int_0^{\frac{1}{A}} \frac{2s^2}{\tilde{r}^3}\left[\frac{\tilde{r}}{s}\,\Phi_{,2}\left(As,\left(\frac{s}{\tilde{r}}\right)^2,\frac{\tilde{r}}{s},\frac{\tilde{r}}{s}\right) - \left(\frac{s}{\tilde{r}}\right)^2\Phi_{,1}\left(As,\left(\frac{s}{\tilde{r}}\right)^2,\frac{\tilde{r}}{s},\frac{\tilde{r}}{s}\right)\right] ds, \tag{1.11.9}$$

where

$$\tilde{r}(R) = (R^3+1)^{\frac{1}{3}}$$

<u>Proposition 1.12</u>

Suppose that Φ satisfies H2 and that

$$\frac{v_1\,\Phi_{,1}(R,v_1,v_2,v_2) - v_2\Phi_{,2}(R,v_1,v_2,v_2)}{v_1 - v_2} \leq A + Bv_2^{\beta} \tag{1.12.1}$$

for $R \in [0,1]$, and $0 < v_1 \leq v_2$, where $A,B > 0$ and $\beta \in (0,2)$ are constants. Then the critical load P^I_{crit} depends only on the material present at the origin, and is the same as the critical load for a homogeneous ball comprised entirely of the material found at the origin.

Proof

The critical load P^I_{crit} (at which bifurcation occurs) is given by the limit as $A \longrightarrow 0$ of (1.11.9). On setting $v = \frac{\bar{r}}{s}$ and comparing with Ball (1982) p.575 we see that this limit formally corresponds to the critical load for a homogeneous ball composed entirely of the material found at the origin. The integrand in (1.11.9) is positive by H2 and on making the change of variable $v = \frac{\bar{r}}{s}$ it takes the form

$$\frac{2}{(v^3-1)v}\left(v\Phi_{,2}(A(v^3-1)^{-\frac{1}{3}}, \frac{1}{v^2}, v, v) - \frac{1}{v^2}\Phi_{,1}(A(v^3-1)^{-\frac{1}{3}}, \frac{1}{v^2}, v, v)\right). \qquad (1.12.2)$$

It follows from (1.12.1) that (1.12.2) is bounded by

$$\frac{2}{(v^3-1)v}\left[A + Bv^{\beta}\right]\left[v - \frac{1}{v^2}\right]$$

it is easily checked that this lies in $L^1(1,\infty)$. This allows us to apply the dominated convergence theorem to pass to the limit $A \longrightarrow 0$ in the integral (1.11.9) (when expressed in the v variable) to

$$P^I_{crit} = \int_1^{\infty} \frac{1}{(v^3-1)} \frac{d}{dv}\left[\Phi(0, \frac{1}{v^2}, v, v)\right] dv.$$

Comparison with Ball (1982) p.575 yields the result.

CHAPTER 2

In this chapter we consider a class of stored energy functions Φ^k of the form

$$\Phi^k(v_1,v_2,v_3) = \Phi^{inc}(v_1,v_2,v_3) + f(k,v_1v_2v_3-1), \tag{2.0.1}$$

where Φ^{inc} represents the stored energy function of an incompressible material and $f : \mathbb{R}^+ \times (-1,\infty) \longrightarrow \mathbb{R}^+$ represents a compressibility term that satisfies

$$f(k,w-1) \longrightarrow \infty \quad \text{as} \quad k \longrightarrow 0, \tag{2.0.2}$$

for each $w \in (0,\infty)$, $w \neq 1$. From (2.0.1) we see that formally, the incompressible stored energy function Φ^{inc} is regained in the limit as $k \longrightarrow 0$. In this limiting process f acts as a 'penalty' term, forcing convergence of the energy minimisers to an incompressible deformation, this is made precise in proposition 2.1. For a general discussion of penalty arguments we refer to Beltrami (1976).

We will examine the behaviour of the critical displacements $\left\{\lambda^k_{crit}\right\}$ and the critical stresses $\left\{P^k_c\right\}$ and $\left\{P^k_{crit}\right\}$ for stored energy functions Φ^k of the form (2.0.1) in the incompressible limit ($k \longrightarrow 0$).

Constitutive Assumptions

We will assume throughout this chapter that the incompressible stored energy function Φ^{inc} has been extended so that $\Phi^{inc} \in C^3(\mathbb{R}^3_{++})$.

We say that the compressibility term $f : \mathbb{R}^+ \times (-1,\infty) \longrightarrow \mathbb{R}^+$ satisfies f1 if there exists a constant $k_o \in (0,\infty)$ such that

(i) $f(k,\cdot) \in C^3((-1,\infty))$ and is convex for each $k \in (0,k_o)$, (2.0.3)

(ii) $f'(k,v) \longrightarrow \infty$ as $v \longrightarrow \infty$ for each $k \in (0,k_o)$, (2.0.4)

where f' denotes differentiation of f with respect to its second argument.

(iii) $f(k,0) \longrightarrow c$ as $k \longrightarrow 0$ where $c \in [0,\infty)$ is a constant, (2.0.5)

(iv) $c_1\frac{|v|}{k} + c_2 \leqslant f(k,v)$ for $k \in (0,k_o)$, where $c_1 > 0$ and $c_2 > 0$ are constants, (2.0.6)

(v) $f(k,v) \longrightarrow \infty$ as $v \longrightarrow -1$ from above for each $k \in (0,k_o)$ (2.0.7)

(vi) $\left|f'(k,v)\right| \leqslant M(\frac{f(k,v)}{v+1} + 1)$ for $k \in (0,k_o)$ and $v \in (-1,\infty)$, (2.0.8)

where $M \in (0,\infty)$ is a constant.

In the course of this chapter we shall draw on a further set of constitutive hypotheses $\Phi1$-$\Phi4$ which are listed together with f1 in the appendix.

We define the admissible set A_λ as in (1.1.1) and the energy I^k corresponding to the stored energy function Φ^k by

$$I^k(r) = \int_0^1 R^2\Phi^k(r', \frac{r}{R}, \frac{r}{R})dR. \tag{2.0.9}$$

Our next proposition relates properties of Φ^k as defined by (2.0.1) to those of Φ^{inc} and f.

Proposition 2.1

Suppose that f satisfies f1 then

(i) If Φ^{inc} satisfies H1 then Φ^k satisfies H1,

(ii) if Φ^{inc} satisfies H2 (respectively H2$^+$) then Φ^k satisfies H2 (respectively H2$^+$),

(iii) if Φ^{inc} satisfies H7 then Φ^k satisfies H7,

(iv) if Φ^{inc} satisfies E2 then Φ^k satisfies E2,

(v) if Φ^{inc} satisfies $\Phi2$ then Φ^k satisfies H5.

Proof

Conditions (i) and (iii) are consequences of (2.0.3) and the definition of Φ^k (2.0.1). (ii) follows immediately from (2.0.1) and (iv) follows from this and (2.0.8).

Since

$$\Phi^k_{,1}(v,a,a) = \Phi^{inc}_{,1}(v,a,a) + a^2 f'(k,va^2-1) \quad \text{for} \quad k \in (0,k_o), \tag{2.1.1}$$

it follows from $\Phi2$ and (2.0.4) that

$$\lim_{v\to\infty} \Phi^k_{,1}(v,a,a) = \infty \quad \text{for} \quad a \in (0,\infty). \tag{2.1.2}$$

By the mean value theorem

$$f(k, \tfrac{1}{n} -1) - f(k,0) = (\tfrac{1}{n} -1)\, f'(k,\theta_n) \quad \text{for} \quad k \in (0,k_o), \tag{2.1.3}$$

for some $\theta_n \in (\frac{1}{n} -1,0)$, for each positive integer n. From (2.1.3) and (2.0.7)

$$f'(k,\theta_n) \longrightarrow -\infty \text{ as } n \longrightarrow \infty \text{ for each } k \in (0,k_o), \tag{2.1.4}$$

and then (2.1.1), $\Phi2$ and (2.1.4) imply that

$\lim_{v \to 0^+} \Phi^k_{,1}(v,a,a) = -\infty$ for each $a \in (0,\infty)$, $k \in (0,k_o)$, which together with (2.1.2) proves (v).

Proposition 2.2

If f satisfies f1 and Φ^{inc} satisfies $\Phi1$, H1 then for each $k \in (0,k_o)$ and each $\lambda \in (0,\infty)$ there exists a minimiser y_k of I^k on A_λ (where I^k is given by (2.0.9)).

Proof

Our hypotheses imply that Φ^k satisfies H1 by proposition 2.1(i). Since f is positive by (2.0.6), we see that Φ^k satisfies the conditions of proposition 1.1, but as we no longer assume that $\Psi(v) \longrightarrow \infty$ as $v \longrightarrow 0$ we must modify the proof accordingly : recall that this condition was used to ensure that any minimiser has strictly positive derivative (except possibly on a set of measure zero). The proof is completed when we note that (2.0.8) guarantees that this still holds.

The next proposition examines the behaviour of minimisers of I^k as $k \longrightarrow 0$.

<u>Proposition 2.3</u>

Suppose that $\dot{\Phi}^{inc}$ satisfies $\Phi 1$, H1, H9, f satisfies f1 and let $\{k_n\}$ be a sequence with $k_n \in (0,k_0)$, $k_n \longrightarrow 0$ as $n \longrightarrow \infty$. If y_{k_n} is a minimiser of I^{k_n} on A_λ then for each $\lambda > 1$

$$y_{k_n} \xrightarrow{W^{1,1}(0,1)} \tilde{r} \quad \text{as} \quad n \longrightarrow \infty, \tag{2.3.1}$$

where

$$\tilde{r}(R) = (R^3 + (\lambda^3 - 1))^{\frac{1}{3}} \tag{2.3.2}$$

is an <u>incompressible</u> deformation.

<u>Proof</u>

The existence of y_{k_n} is a consequence of proposition 2.2. It then follows that

$$I^{k_n}(y_{k_n}) = \inf_{y \in A_\lambda} I^{k_n}(y) \leq I^{k_n}(\tilde{r}) = \int_0^1 R^2 \Phi^{inc}(\tilde{r}', \frac{\tilde{r}}{R}, \frac{\tilde{r}}{R}) dR + \int_0^1 R^2 f(k_n, 0) dR \tag{2.3.3}$$

for all n. Setting $v = \frac{\tilde{r}(R)}{R}$ in the right hand side expression in (2.3.3) and using H9 we obtain

$$\int_0^1 R^2 \Phi(\tilde{r}', \frac{\tilde{r}}{R}, \frac{\tilde{r}}{R}) dR = A^3 \int_\lambda^{\infty} \frac{v^2}{v^3 - 1} \dot{\Phi}^{inc}(v) \, dv < +\infty ,$$

where $A = (\lambda^3 - 1)^{\frac{1}{3}}$, and hence (2.0.5) and (2.3.3) imply that

$$I^{k_n}(y_{k_n}) < \text{constant uniformly in } n. \tag{2.3.4}$$

We now make the change of variables

$$u_{k_n} = y_{k_n}^3 \quad \text{and} \quad \rho = R^3$$

in (2.3.4). Then writing $\dot{u}_{k_n}$ for $\frac{du_{k_n}}{d\rho}$ we obtain

$$\dot{u}_{k_n} = \left(\frac{y_{k_n}}{R}\right)^2 y'_{k_n} . \tag{2.3.5}$$

From (2.3.4), (2.3.5) and (2.0.6) we obtain

$$\int_0^1 |\dot{u}_{k_n} - 1| \, d\rho \leq c.k_n \quad \text{for all } n \tag{2.3.6}$$

where c is a constant. Then since $u_{k_n}(1) = \lambda^3$

$$u_{k_n} \xrightarrow{W^{1,1}(0,1)} \tilde{u} \quad \text{as } n \longrightarrow \infty, \tag{2.3.7}$$

where $\tilde{u}(\rho) = (\rho + (\lambda^3 - 1))$. As $\lambda > 1$, it follows from (2.3.7) that $u_{k_n}(0)$ is uniformly bounded away from zero. Hence by (2.3.2) and (2.3.5) we obtain

$$\int_0^1 R^2 \left| \left(\frac{y_{k_n}}{R}\right)^2 y'_{k_n} - \left(\frac{\tilde{r}}{R}\right)^2 \tilde{r}' \right| dR = \int_0^1 |\dot{u}_{k_n} - \dot{\tilde{u}}| d\rho \tag{2.3.8}$$

and that

$$\int_0^1 | y'_{k_n} - \tilde{r}' | \, dR \leq \text{const.} \left[\int_0^1 \left| y_{k_n}^2 y_{k_n}' - y_{k_n}^2 \tilde{r}' \right| dR \right] \tag{2.3.9}$$

for all n. Since

$$\left| y_{k_n}^2 y_{k_n}' - y_{k_n}^2 \tilde{r}' \right| \leq \left| y_{k_n}^2 y_{k_n}' - \tilde{r}^2 \tilde{r}' \right| + \left| \tilde{r}^2 \tilde{r}' - y_{k_n}^2 \tilde{r}' \right| \tag{2.3.10}$$

for all n by the triangle inequality and as the second term on the right hand side of (2.3.10) is bounded by $2\lambda^2 \tilde{r}' \in L^1(0,1)$, it follows from (2.3.7) - (2.3.10) and the dominated convergence theorem that

$$y'_{k_n} \xrightarrow{L^1(0,1)} \tilde{r}' \quad \text{as } n \longrightarrow \infty .$$

Finally, as $y_{k_n}(1) = \lambda$ for all n (2.3.1) holds.

We require that $\Phi^k_{,i}(1,1,1) = 0$ for $i = 1,2,3$, for each $k \in (0,k_0)$ so that the reference configuration is a natural state.

Proposition 2.4

Suppose that Φ^{inc} satisfies H1, H2$^+$, H7, H9, Φ1 and Φ2 and f satisfies f1. Then for each $k \in (0,k_0)$ there exists a critical value $\lambda^k_{crit} > 1$ such that the minimiser y_k of I^k on A_λ satisfies $y_k(0) = \lim_{R \to 0} y_k(R) > 0$ if and only if $\lambda > \lambda^k_{crit}$. Moreover $y_k \in C^2((0,1])$ for all $\lambda \in (0,\infty)$ and if $\lambda < \lambda^k_{crit}$ then $y_k(R) \equiv \lambda R$.

Proof

It follows from proposition 1.8, (2.0.4) and remark 1.9 that a cavitating minimiser exists for sufficiently large λ . By proposition 2.1 and proposition 0.3 it is smooth and satisfies the equilibrium equation. The proof is completed on applying corollary 0.15.

Theorem 2.5

Let f satisfy f1, let Φ^{inc} satisfy H1, H2$^+$, H7, H9, Φ1 and Φ2. If λ^k_{crit} is the critical displacement corresponding to the stored energy function Φ^k then

$$\lambda^k_{crit} \longrightarrow 1 \quad \text{as} \quad k \longrightarrow 0. \tag{2.5.1}$$

Proof

The existence of λ^k_{crit} is a consequence of proposition 2.4. We suppose for a contradiction that (2.5.1) does not hold, then by proposition 2.4 there exist a sequence $\{k_n\} \longrightarrow 0$ and $\partial_0 > 1$ satisfying

$$\lambda^{k_n}_{crit} \geq \partial_0 > 1 \quad \text{for all } n. \tag{2.5.2}$$

Fixing $\lambda \in (1,\partial_o)$ and applying proposition 2.3 and 2.4 we obtain

$$y_{k_n} = y \xrightarrow{W^{1,1}(0,1)} \tilde{r} \quad \text{as} \quad n \longrightarrow \infty,$$

where $y(R) \equiv \lambda R$ and $\tilde{r}(R)$ is given by (2.3.2), which is clearly false.

In the rest of this chapter we prove a similar convergence result for the critical stresses (recall that for a compressible stored energy function Φ with corresponding critical displacement λ_{crit} the critical Cauchy and Piola stresses P_{crit} and P_c are given by

$$\frac{1}{(\lambda_{crit})^2} \Phi_{,1}(\lambda_{crit},\lambda_{crit},\lambda_{crit}) \text{ and } \Phi_{,1}(\lambda_{crit},\lambda_{crit},\lambda_{crit}) \tag{2.5.3}$$

respectively). The next proposition is central to our arguments.

Proposition 2.6

Suppose f satisfies f1 and Φ^{inc} satisfies H1, H2 and Φ2. Let $r_k \in C^2((0,1])$ be a cavitating equilibrium solution corresponding to the stored energy function Φ^k with $k \in (0,k_o)$, then r_k is extendable to $r_k \in C^2((0,\infty))$ as a solution of (0.18) and $T_k(r_k(\cdot))$ is absolutely continuous on $[0,\infty]$, where

$$T_k(r_k(R)) \overset{def}{=} \left(\frac{R}{r_k(R)}\right)^2 \Phi^k_{,1}\left(r_k'(R), \frac{r_k(R)}{R}, \frac{r_k(R)}{R}\right) \tag{2.6.1}$$

Proof

It is a consequence of proposition 2.1 and proposition 0.9 that r_k is extendable to $r_k \in C^2((0,\infty))$ as a solution of (0.2.3), hence

$$\frac{dT_k(r_k(R))}{dR} = \frac{2R^2}{(r_k(R))^3}\left(\frac{r_k(R)}{R} \Phi^k_{,2}(R) - r_k'(R)\Phi^k_{,1}(R)\right) \tag{2.6.2}$$

for $R \in (0,\infty)$. Clearly $\frac{dT_k(r_k(R))}{dR} \in L^1(\partial,M)$ for $0 < \partial < M < +\infty$.

By proposition 2.1, corollary 0.5 and proposition 0.6 it follows that

$\frac{d}{dR} T_k(r_k(R)) > 0$ for $R \in (0,\infty)$. Finally, since $\lim_{R \to 0} T_k(r_k(R)) = 0$,

it follows from proposition 0.9 and the monotone convergence theorem that

$\frac{d}{dR} T_k(r_k(R)) \in L^1(0,\infty)$.

It is an immediate consequence of proposition 0.9, (2.6.1) and (2.5.3) that

$$P^k_{crit} = \lim_{R \to \infty} T_k(r_k(R)) \quad \text{for each} \quad k \in (0,k_o). \tag{2.6.3}$$

The key to resolving our convergence problem lies in the observation that

$$P^k_{crit} = \int_0^\infty \frac{dT_k(r_k(s))}{ds}\, ds = \int_0^\infty \frac{2s^2}{(r_k(s))^3}\left(\frac{r_k(s)}{s}\,\Phi^k_{,2}(s) - r_k'(s)\Phi^k_{,1}(s)\right)ds \tag{2.6.4}$$

$$= \int_0^\infty \frac{2s^2}{(r_k(s))^3}\left(\frac{r_k(s)}{s}\,\Phi^{inc}_{,2}(s) - r_k'(s)\Phi^{inc}_{,1}(s)\right)ds. \tag{2.6.5}$$

Equation (2.6.4) is a consequence of (2.6.3), proposition 2.5 and the fact that $T_k(r_k(0)) = 0$. Equation (2.6.5) then follows immediately from (2.0.1).

From Ball (1982) p.575 we note that if Φ^{inc} is an <u>incompressible</u> stored energy function satisfying $\Phi3$ then the critical Cauchy stress P^{inc}_{crit} is given by

$$P^{inc}_{crit} = \int_1^\infty \frac{1}{v^3-1}\,\frac{d\hat{\Phi}^{inc}}{dv}(v)\, dv = \int_1^\infty \frac{1}{(v^3-1)}\,\frac{2}{v}\left(v\hat{\Phi}^{inc}_{,2}(v) - \frac{1}{v^2}\,\hat{\Phi}^{inc}_{,1}(v)\right)dv \tag{2.6.6}$$

We next set $v = \frac{\tilde{r}(s)}{s}$ in (2.6.6), where $\tilde{r}$ is defined by (2.3.2). Upon noting that $\tilde{r}'(s) = \left(\frac{s}{\tilde{r}(s)}\right)^2$, (2.6.6) takes the form

$$P^{inc}_{crit} = \int_0^\infty \frac{2s^2}{(\tilde{r}(s))^3}\left(\frac{\tilde{r}(s)}{s}\,\Phi^{inc}_{,2}(s) - \tilde{r}'(s)\Phi^{inc}_{,1}(s)\right)ds. \tag{2.6.7}$$

Clearly the convergence of the critical Cauchy stresses P^k_{crit} to P^{inc}_{crit} is proved if we can pass to the limit $k \longrightarrow 0$ from (2.6.5) to (2.6.7). However, this presents some technical difficulties and the rest of this chapter is geared towards overcoming them.

Proposition 2.7

Let f satisfy f1 and let Φ^{inc} satisfy H1, H2, H7, H9, Φ1 and Φ2. Then for each $k \in (0,k_o)$ there exists $g_k : (\lambda^k_{crit},\infty) \longrightarrow (0,\infty)$, where $g_k \in C^1((\lambda^k_{crit},\infty))$ and satisfies

$$(g_k(w)-w)\frac{d}{dw}(\Phi^k_{,1}(g_k(w),w,w)) = 2(\Phi^k_{,2}(g_k(w),w,w)-\Phi^k_{,1}(g_k(w),w,w)) \qquad (2.7.1)$$

for $w \in (\lambda^k_{crit},\infty)$. Moreover, if $\lambda \in (\lambda^k_{crit},\infty)$ and r_k is the minimiser of I^k on A_λ then

$$g_k\left(\frac{r_k(R)}{R}\right) = r_k'(R) \quad \text{for} \quad R \in (0,1]. \qquad (2.7.2)$$

Proof

The existence of λ^k_{crit} is a consequence of proposition 2.4. It also follows from proposition 2.4 that if $\lambda \in (\lambda_{crit},\infty)$ then the minimiser r_k of I^k on A_λ satisfies $r_k(0) > 0$. By proposition 2.1 Φ^k satisfies the conditions of corollary 0.15 hence r_k is extendable to $r_k \in C^2((0,\infty))$ as a solution of (0.18) and satisfies $\lim_{R\to\infty} \frac{r_k(R)}{R} = \lambda^k_{crit}$. (2.7.3)

We now set

$$w = \frac{r_k(R)}{R}, \qquad (2.7.4)$$

$$g_k = r_k' . \qquad (2.7.5)$$

From corollary 0.5 and the inverse function theorem it follows that (2.7.4) may be inverted to give R as a C^1 function of w, hence $g_k \in C^1((\lambda^k_{crit},\infty))$ and

$$R\frac{d}{dR} = R\frac{dw}{dR} = (g_k(w)-w)\frac{d}{dw}\,. \tag{2.7.6}$$

Using (2.7.6) the equilibrium equation (0.2.3) takes the form (2.7.1) as required.

By construction $g_k(\frac{r_k(R)}{R}) = r_k'(R)$ for $R \in (0,\infty)$; part (iv) of theorem 0.14 then implies that the validity of (2.7.2) is independent of the choice of $\lambda \in (\lambda_{crit},\infty)$.

Proposition 2.8

If f satisfies f1 and Φ^{inc} satisfies H1, H2, H7, H9, Φ1-Φ4 then

$$g_k(w) \longrightarrow \frac{1}{w^2} \quad \text{as} \quad k \longrightarrow 0 \tag{2.8.1}$$

for each $w \in (1,\infty)$, where g_k is defined as in proposition 2.6.

Proof

The proof proceeds in two stages; we show first that

$$\Phi^k_{,1}(g_k(w),w,w) \longrightarrow \text{constant} \quad \text{as} \quad k \longrightarrow 0,$$

for each $w \in (1,\infty)$ and then that this and our assumptions on the structure of Φ^k together imply that (2.8.1) holds.

Step 1

Fix $w_o \in (1,\infty)$, then by theorem 2.5 there exists a constant $c_o \in (0,\infty)$ such that

$$\lambda^k_{crit} < w_o \quad \text{for} \quad k \in (0,c_o), \tag{2.8.2}$$

and so $g_k(w_o)$ is well defined. Let $\{k_n\}$, $k_n \in (0,c_o)$ be a sequence with $k_n \longrightarrow 0$ as $n \longrightarrow \infty$. Then applying proposition 2.3 with $\lambda = w_o$ we may assume without loss of generality that the minimisers r_{k_n} of I^{k_n} on A_{w_o} satisfy

$$r_{k_n}'(R) \longrightarrow \tilde{r}'(R) = \left(\frac{R}{\tilde{r}(R)}\right)^2 \quad \text{as } n \longrightarrow \infty \text{ pointwise for a.e. } R \in (0,1) \tag{2.8.3}$$

and

$$\frac{r_{k_n}(R)}{R} \longrightarrow \frac{\tilde{r}(R)}{R} \quad \text{as} \quad n \longrightarrow \infty \text{ pointwise for a.e. } R \in (0,1), \tag{2.8.4}$$

$$\text{where} \quad \tilde{r}(R) = (R^3+w_o^3-1)^{\frac{1}{3}}. \tag{2.8.5}$$

By (2.8.2), proposition 2.4 and proposition 0.3 the r_{k_n} are cavitating equilibrium solutions. It then follows from (2.7.2) and proposition 2.6 that

$$\frac{1}{w_o^2}\,\Phi^{k_n}_{,1}(g_{k_n}(w_o),w_o,w_o) = \frac{1}{(r_{k_n}(1))^2}\,\Phi^{k_n}_{,1}(r_{k_n}'(1),r_{k_n}(1),r_{k_n}(1))$$

$$= \int_0^1 \frac{d}{dR}\left[\left(\frac{R}{r_{k_n}}\right)^2 \Phi^{k}_{,1}\left(r_{k_n}', \frac{r_{k_n}}{R}, \frac{r_{k_n}}{R}\right)\right] dR \tag{2.8.6}$$

for each n, where we have incorporated the zero stress boundary condition. By (2.6.2) and (2.0.1), (2.8.6) takes the form

$$\int_0^1 \frac{2R^2}{(r_{k_n})^3}\left[\frac{r_{k_n}}{R}\,\Phi^{inc}_{,2}(R) - r_{k_n}'\Phi^{inc}_{,1}(R)\right] dR \tag{2.8.7}$$

for each n. Corollary 0.5 and proposition 0.6 together imply that the integrand in (2.8.7) is positive, hence, using $\Phi 4$ we obtain

$$0 \leq \frac{2R^2}{(r_{k_n})^3}\left(\frac{r_{k_n}}{R}\,\Phi^{inc}_{,2} - r_{k_n}'\Phi^{inc}_{,1}\right) \leq \frac{2R^2}{(r_{k_n})^3}\left(\frac{r_{k_n}}{R} - r_{k_n}'\right)\left(A + B\left(\frac{r_{k_n}}{R}\right)^{\beta}\right)$$

$$\leq \frac{4R}{(r_{k_n})^2}\left(A + B\left(\frac{r_{k_n}}{R}\right)^{\beta}\right) \leq C_1 + C_2\,\frac{R^{1-\beta}}{(r_{k_n})^{2-\beta}} \leq C_1 + C_3 R^{1-\beta} \tag{2.8.8}$$

for all n, where C_1, C_2 and C_3 are constants. To obtain (2.8.8) we have used corollary 0.5 and the fact that $r_{k_n}(0)$ is uniformly bounded away from zero by proposition 2.3. Since $\beta < 2$ the right hand side of (2.8.8) is in $L^1(0,1)$ and so by (2.8.3), (2.8.4) and the dominated convergence theorem we can pass to the limit in (2.8.6) and (2.8.7) to obtain

$$\frac{1}{w_o^2}\,\Phi^{k}_{,1}(g_{k_n}(w_o),w_o,w_o) \longrightarrow \int_0^1 \frac{2R^2}{(\tilde{r})^2}\left(\frac{\tilde{r}}{R}\,\Phi^{inc}_{,2}(R) - \tilde{r}'\Phi^{inc}_{,1}(R)\right) dR \qquad (2.8.9)$$

as $n \longrightarrow \infty$, where r is defined by (2.8.5). Setting $v = \frac{\tilde{r}(R)}{R}$ now gives the integral in (2.8.9) the form

$$\int_{w_o}^{\infty} \frac{1}{v^3-1}\,\frac{d\Phi^{inc}}{dv}(v)\,dv. \qquad (2.8.10)$$

The expression (2.8.10) is finite by $\Phi 3$. Hence

$$\frac{1}{w_o^2}\,\Phi^{k_n}_{,1}(g_{k_n}(w_o),w_o,w_o) \longrightarrow \text{constant} \quad \text{as} \quad n \longrightarrow \infty. \qquad (2.8.11)$$

Step 2

From (2.8.11) and the definition of Φ^k it follows that

$$\frac{1}{w_o^2}\,\Phi^{inc}_{,1}(g_{k_n}(w_o),w_o,w_o) + f'(k_n, g_{k_n} w_o^2 - 1) \longrightarrow \text{constant} \quad \text{as} \quad n \longrightarrow \infty. \qquad (2.8.12)$$

We now suppose for a contradiction that (2.8.1) does not hold so that without loss of generality there exist $w \in (1,\infty)$, $\partial_o > 0$ and a sequence $\{k_j\}$ converging to zero satisfying either

(i) $g_{k_j}(w)w^2 < 1 - \partial_o$ for all j, or (2.8.13)

(ii) $g_{k}(w)w^2 > 1 + \partial_o$ for all j. (2.8.14)

If (2.8.13) holds then by the convexity of f ((2.0.3)) for each j

$$f'(k_j, g_{k_j}(w)w^2 - 1) \leqslant f'(k_j, -\partial_o). \qquad (2.8.15)$$

But

$$f(k_j,-\partial_o) - f(k_j,0) = -\partial_o f'(k_j,\theta_j) \quad \text{for each } j, \qquad (2.8.16)$$

for some $\theta_j \in (-\partial_o, 0)$. It then follows from (2.8.16), (2.0.5) and (2.0.6) that

$$f'(k_j,\theta_j) \longrightarrow -\infty \text{ as } j \longrightarrow \infty.$$

and hence by the convexity of f

$$f'(k_j,-\partial_o) \longrightarrow -\infty \text{ as } j \longrightarrow \infty . \tag{2.8.17}$$

(2.8.17), (2.8.15) and $\Phi 2$ together contradict (2.8.12). A similar contradiction is obtained if (2.8.14) holds on using (2.0.4) and $\Phi 2$.

Theorem 2.9

Let f satisfy f1 and let Φ^{inc} satisfy H1, H2, H7, H9, $\Phi 1$-$\Phi 4$. If P^k_{crit} are the critical Cauchy stresses corresponding to the stored energy function Φ^k then

$$P^k_{crit} \longrightarrow P^{inc}_{crit} \text{ as } k \longrightarrow 0, \tag{2.9.1}$$

where P^{inc}_{crit} is defined by (2.6.6).

Proof

Setting $w = \frac{r_k(s)}{s}$ in (2.6.4) and (2.6.5) we obtain

$$P^k_{crit} = \int_{\infty}^{\lambda^k_{crit}} \frac{d\tilde{T}_k(w)}{dw}\, dw = \int_{\lambda^k_{crit}}^{\infty} 2\, \frac{(w\Phi^{inc}_{,2}(g_k(w),w,w) - g_k(w)\Phi^{inc}_{,1}(g_k(w),w,w))}{w^3(w-g_k(w))}\, dw \tag{2.9.2}$$

where $\tilde{T}_k(w) \overset{def}{=} \frac{1}{w^2}\, \Phi^k_{,1}(g_k(w),w,w)$ and g_k is defined as in proposition 2.7.

As Φ^{inc} satisfies $\Phi 4$ the integrand in (2.9.2) is bounded by $\frac{1}{w^3}(A+Bw^\beta)$. Since $\beta < 2$ this lies in $L^1(1,\infty)$. Line (2.9.1) then follows on application of proposition 2.8 and the dominated convergence theorem.

Corollary 2.10

The critical Piola stresses P^k_c satisfy

$$P^k_c \longrightarrow P^{inc}_{crit} \text{ as } k \longrightarrow 0.$$

Proof

This is an immediate consequence of (2.9.1) and theorem 2.9 on noting that

$$P_c^k = (\lambda_{crit}^k)^2 P_{crit}^k.$$

CHAPTER 3

In section 1, using phase plane techniques, we prove the uniqueness of solutions to the displacement and mixed displacement/traction boundary value problems for 'punctured' balls of internal radius ε. The proofs consist of showing that an appropriate 'time map' is monotone, and rely on the change of variables

$$v = \frac{r}{R}, \quad e^s = R, \tag{3.0.1}$$

which gives (0.2.3) the autonomous form

$$\frac{d}{ds}\left[\Phi_{,1}(\dot{v}+v,v,v)\right] = 2\left[\Phi_{,2}(\dot{v}+v,v,v) - \Phi_{,1}(\dot{v}+v,v,v)\right]. \tag{3.0.2}$$

The results of section 1 motivate a change of variables in the energy functional, which is used in section 2 to prove the uniqueness of cavitating equilibrium solutions. The chapter is concluded with a proof of theorem 0.14.

1. Punctured Balls

We define a punctured ball B^ε of internal radius by

$$B^\varepsilon = \left\{ \underline{X} \in \mathbb{R}^3;\ \varepsilon < |\underline{X}| < 1 \right\}. \tag{3.0.3}$$

Correspondingly we define a radial equilibrium solution to the mixed displacement/traction problem to be any solution $r_\varepsilon \in C^2([\varepsilon,1])$ of (0.2.3) satisfying

(i) $r_\varepsilon'(R) > 0$ for $R \in \varepsilon,1$, (3.0.4)

(ii) $r_\varepsilon(1) = \lambda$ (3.0.5)

(iii) $r_\varepsilon(\varepsilon) > 0$ (3.0.6)

(iv) $\lim_{R \to \varepsilon} T(r_\varepsilon(R)) = 0$ (3.0.7)

In condition (ii) $\lambda > 0$ is the boundary displacement, and (iv) is the natural boundary condition that the cavity is stress free.

We first give conditions on the stored energy function which guarantee that the points of zero stress form a well defined curve in phase space; crossing the $\dot{v} = 0$ axis with negative slope.

Proposition 3.1

If Φ satisfies H1 and H5 then there exists a unique function $\sigma \in C^1((0,\infty))$ satisfying

$$\Phi_{,1}(\sigma(v)+v,v,v) = 0 \quad \text{for all} \quad v \in (0,\infty) \tag{3.1.1}$$

If in addition Φ satisfies H6 then

$$\sigma'(1) > 0. \tag{3.1.2}$$

Proof

It follows from H1 and H5 that for each $v_o \in (0,\infty)$ there exists a unique $\alpha(v_o) \in (-v_o,\infty)$ such that $\Phi_{,1}(\alpha(v_o)+v_o,v_o,v_o) = 0$. The existence of $\sigma \in C^1((0,\infty))$ satisfying (3.1.1) is then a consequence of H1 and the implicit function theorem. Implicit differentiation of (3.1.1) with respect to v gives

$$(\sigma'(v)+1)\ \Phi_{,11}(\sigma(v)+v,v,v) + 2\Phi_{,12}(\sigma(v)+v,v,v) = 0 \tag{3.1.3}$$

and hence

$$\sigma'(v) = \frac{-2\Phi_{,12}(\sigma(v)+v,v,v)}{\Phi_{,11}(\sigma(v)+v,v,v)} - 1. \tag{3.1.4}$$

Hypothesis H6 is the condition that

$$\det\ (\Phi_{,ij}(v_1,v_2,v_3))\Big|_{v_i=1} > 0\ . \tag{3.1.5}$$

On setting $X = \dfrac{\Phi_{,12}(1,1,1)}{\Phi_{,11}(1,1,1)}$, (3.1.5) takes the form

$$1 - 3X^2 + 2X^3 = (X-1)^2(2X+1) > 0.$$

and hence $X > \frac{-1}{2}$. Condition (3.1.2) then follows from (3.1.4) and the definition of X.

Corollary 3.2

The functions $\Phi_{,1}(\lambda,\lambda,\lambda)$ and $\frac{1}{\lambda^2}\Phi_{,1}(\lambda,\lambda,\lambda)$ are monotone in a neighbourhood of $\lambda = 1$.

Proof

It is easily seen that

$$\frac{d}{d\lambda}\left[\frac{1}{\lambda^2}\Phi_{,1}(\lambda,\lambda,\lambda)\right]\Bigg|_{\lambda=1} = \frac{\Phi_{,11}(\lambda)}{\lambda^2}\left[1+\frac{2\Phi_{,12}(\lambda)}{\Phi_{,11}(\lambda)}\right]\Bigg|_{\lambda=1}, \tag{3.2.1}$$

when we use the fact that the undeformed configuration is a natural state. The result follows on noting that the right hand side of (3.2.1) is strictly positive by proposition 3.1(ii) and (3.1.4). A similar argument applies in the case of $\Phi_{,1}(\lambda,\lambda,\lambda)$.

Remark 3.3

Notice that $v \equiv$ constant is always a solution of (3.0.2). Hence the v-axis is a line of rest points and consideration of the phase portrait then shows that any non constant C^2 solution $v(s)$ of (3.0.2) satisfies one of the two following conditions

(i) $\dot{v}(s) > 0$ for all s in the interval of existence or

(ii) $\dot{v}(s) < 0$ for all s in the interval of existence.

Hence

$$\left[\frac{d}{ds}\dot{v}(s)\right]\frac{1}{\dot{v}(s)} = \frac{d\dot{v}}{dv} \doteq 2\left[\frac{\Phi_{,2}(\dot{v}+v,v,v)-\Phi_{,1}(\dot{v}+v,v,v)}{\dot{v}\,\Phi_{,11}(\dot{v}+v,v,v)} - \frac{\Phi_{,12}(\dot{v}+v,v,v)}{\Phi_{,11}(\dot{v}+v,v,v)}\right] - 1 \tag{3.3.1}$$

$$= 2\left[\frac{\int_0^1\left[\Phi_{,21}(t\dot{v}+v,v,v) - \Phi_{,11}(t\dot{v}+v,v,v)\right]dt - \Phi_{,12}(\dot{v}+v,v,v)}{\Phi_{,11}(\dot{v}+v,v,v)}\right] - 1 \tag{3.3.2}$$

$$\stackrel{def}{=} G(v,\dot{v}) \tag{3.3.3}$$

and

$$G \in C^1(H) \quad \text{where } H = \left\{ (v,\dot{v}) \in \mathbb{R}^2;\ v > 0,\ \dot{v}+v > 0 \right\} . \tag{3.3.4}$$

It then follows that solutions $v(s)$ of (3.0.2) generate solutions of

$$\frac{d\dot{v}}{dv} = G(v,\dot{v}) \tag{3.3.5}$$

and conversely solution curves of (3.3.5) are invariant manifolds for the flow generated by (3.0.2).

If Φ satisfies H1 and H5, then the points of zero stress lie on a curve $\dot{v} = \sigma(v)$ in phase space by proposition 3.1. Moreover if $\dot{v} = f_\partial(v)$ is a C^1 solution of (3.3.5) on an interval containing the points $\lambda > 0$, $\partial > 0$ satisfying

(i) $f_\partial(\partial) = \sigma(\partial)$ and

(ii) $f_\partial(v) \neq 0$ for $v \in [\partial,\lambda]$ (or $v \in [\lambda,\partial]$)

where ∂ and λ are positive constants, then we define the <u>time</u> <u>map</u> T by

$$T(\partial) = \int_\partial^\lambda \frac{1}{f_\partial(v)}\, dv . \tag{3.3.6}$$

Our next theorem concerns the uniqueness of equilibrium solutions to the mixed problem for punctured balls of internal radius ε and is one of the main results of this section.

<u>Theorem 3.4</u>

Let Φ satisfy H1, H2, H5. Then for each $\varepsilon \in (0,1)$ and $\lambda \in (0,\infty)$ there exists at most one solution $r_\varepsilon \in C^2([\varepsilon,1])$ of (0.2.3) satisfying (3.0.4) - (3.0.6).

<u>Proof</u>

The proof proceeds in 3 stages; first we characterise the phase portraits corresponding to r_ε, secondly we prove a monotonicity property associated with the time map T and finally we show that this monotonicity implies the uniqueness of r_ε.

Step 1

Fix $\varepsilon \in (0,1)$ and let $r \in C^2([\varepsilon,1])$ be a non trivial solution of (0.2.3) which satisfies (3.0.4) - (3.0.6). Then under the change of variables given by (3.0.1) $r(R)$ gives rise to a non constant solution $v(s)$ of (3.0.2) where $v \in C^2([\log\varepsilon,0])$ and satisfies

(i) $v(0) = \lambda$. (3.4.1)

(ii) $\dot{v}(s)+v(s)>0$ for $s\in [\log\varepsilon,0]$, (3.4.2)

(iii) $\Phi,_1(\dot{v}+v,v,v)\Big|_{s=\log\varepsilon} = 0.$ (3.4.3)

We claim that $v(s)$ satisfies one of the two following conditions; either

(a) $\sigma(v(s)) \leqslant \dot{v}(s) < 0$ for $s \in [\log\varepsilon,0]$ or (3.4.4)

(b) $\sigma(v(s)) \geqslant \dot{v}(s) \geqslant 0$ for $s\in [\log\varepsilon,0]$. (3.4.5)

The arguments contained in remark 3.3 imply that $\dot{v}(s)$ is single signed for $s \in [\log\varepsilon,0]$. We suppose that $\dot{v}(s) < 0$ for $s\in[\log\varepsilon,0]$; then proposition 0.6, (3.1.1) and (3.0.1) together imply that

$$\frac{1}{v^2(s)}\,\Phi,_1(\dot{v}(s)+v(s),v(s),v(s)) \geqslant 0 = \frac{1}{v^2(s)}\,\Phi,_1(\sigma(v(s))+v(s),v(s),v(s)) \tag{3.4.6}$$

for $s \in [\log\varepsilon,0]$. Hence (3.4.4) follows from H1. A similar proof holds for (3.4.5) in the case $\dot{v}(s) > 0$ for $s\in [\log\varepsilon,0]$. To justify our consideration of non constant solutions $v(s)$ we make the following remark; if $v(s) \equiv \lambda$ satisfies (3.4.3) then $\sigma(\lambda) = 0$. Condition (3.4.4) or (3.4.5) evaluated at $s = 0$ together with H1 then imply that this constant solution is unique amongst all solutions of (3.0.2) satisfying (3.4.1) - (3.4.3) and the theorem holds.

Step 2

Let $f_{\partial_i} \in C^1([\lambda,\partial_i])$ $i = 1,2$ be two distinct solutions of (3.3.5) satisfying

$$\sigma(\partial_i) = f_{\partial_i}(\partial_i) \quad i = 1,2 \; . \tag{3.4.8}$$

We claim that if

$$\text{(i)} \quad \sigma(v) \leq f_{\partial_i}(v) < 0 \quad \text{for } v \in [\lambda,\partial_i] \quad i = 1,2 \; , \tag{3.4.9}$$

where ∂_i are positive constants with $\lambda < \partial_1 < \partial_2$,

or if

$$\text{(ii)} \; 0 < f_{\partial_i}(v) \leq \sigma(v) \quad \text{for } v \in [\lambda,\partial_i] \quad i = 1,2 \; , \tag{3.4.10}$$

where ∂_i are positive constants with $\partial_2 < \partial_1 < \lambda$,

then

$$T(\partial_1) < T(\partial_2). \tag{3.4.13}$$

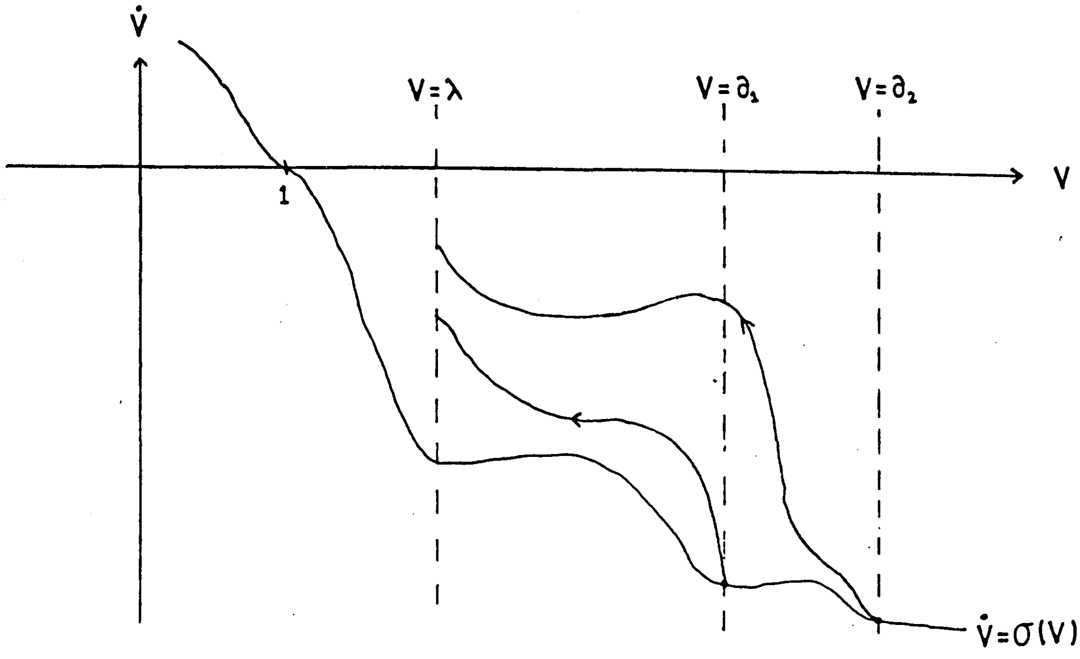
We prove (i) (the proof of (ii) is identical in nature and will be omitted). Uniqueness of solutions to the initial value problem for (3.3.5) implies that

$$f_{\partial_1}(v) < f_{\partial_2}(v) \quad \text{for } v \in [\lambda,\partial_1] \; . \tag{3.4.14}$$

Using the definition of the time map (3.3.6) we obtain

$$T(\partial_1) = -\int_{\lambda}^{\partial_1} \frac{1}{f_{\partial_1}(v)}\,dv < -\int_{\lambda}^{\partial_2} \frac{1}{f_{\partial_2}(v)}\,dv = T(\partial_2) \tag{3.4.15}$$

from (3.4.9) and (3.4.14) and hence (3.4.13) holds.



Step 3

Now let $v \in C^2([\log \varepsilon, 0])$ be a solution of (3.0.2) that satisfies (3.4.1), (3.4.2) and (3.4.4) (an exactly analogous argument holds in the case of (3.4.5)). The arguments contained in remark 3.3 imply that $v(s)$ generates a solution $f_\partial \in C^1([\lambda, \partial])$ of (3.3.5) satisfying $f_\partial(\partial) = \sigma(\partial)$, where $\partial = v(\log \varepsilon)$. It then follows that

$$\frac{1}{f_\partial(v(s))} \frac{dv(s)}{ds} = 1 \quad \text{for } s \in [\log \varepsilon, 0] \tag{3.4.16}$$

and so

$$T(\partial) = \int_\partial^\lambda \frac{1}{f_\partial(v)}\, dv = \int_\partial^\lambda \frac{1}{f_\partial(v(s))} \frac{dv(s)}{ds}\, ds = \int_{\log \varepsilon}^0 1\, ds = \log \frac{1}{\varepsilon} \quad . \tag{3.4.17}$$

The proof of the theorem is completed on noting that by (3.4.4) and (3.4.5) any two distinct solutions $v_i(s)$ $i = 1,2$ of (3.0.2) that satisfy (3.4.1) - (3.4.3) will generate two distinct functions f_{∂_i} $i = 1,2$ satisfying the conditions (3.4.8) and (3.4.9) (or (3.4.10)) of Step 2; (3.4.13) and (3.4.17) then yield a contradiction.

Our next result concerns the uniqueness of solutions to the displacement boundary value problem for a punctured ball of internal radius $\varepsilon \in (0,1)$; equilibrium configurations for this problem correspond to solutions $r_\varepsilon \in C^2([\varepsilon, 1])$ of (0.2.3) that satisfy

(i) $r_\varepsilon(1) = \lambda,$ (3.4.18)

(ii) $r_\varepsilon'(R) > 0$ for $R \in [\varepsilon, 1]$ and (3.4.19)

(iii) $r_\varepsilon(\varepsilon) = \mu,$ (3.4.20)

where λ and μ are given constants with $0 < \mu < \lambda$.

Theorem 3.5

Suppose that Φ satisfies H1. Then for each $\varepsilon \in (0,1)$ there exists at most one solution $r_\varepsilon \in C^2([\varepsilon, 1])$ of (0.2.3) satisfying (3.4.18) - (3.4.20).

Proof

We proceed in an analogous manner to the proof of theorem 3.4.

Step 1

Fix $\varepsilon \in (0,1)$; then any solution $r \in C^2([\varepsilon, 1])$ that satisfies (3.4.18) - (3.4.20) generates a solution $V \in C^2([\log\varepsilon, 1])$ of (3.0.2) satisfying

(i) $v(0) = \lambda,$ (3.5.1)

(ii) $\dot{v}(s) + v(s) > 0$ for $s \in [\log\varepsilon, 0]$, (3.5.2)

(iii) $v(\log\varepsilon) = \frac{\mu}{\varepsilon}$. (3.5.3)

We assume without loss of generality that $\frac{\mu}{\varepsilon} < \lambda$.

Step 2

If $f_\partial \in C^1([\frac{\mu}{\varepsilon}, \lambda])$ is a solution of (3.3.5) satisfying

(i) $f_\partial(v) > 0$ for $v \in [\frac{\mu}{\varepsilon}, \lambda]$, (3.5.4)

(ii) $f_\partial(\frac{\mu}{\varepsilon}) = \partial$, (3.5.5)

where $\partial > 0$ is a constant, then we define the time map T^* by

$$T^*(\partial) = \int_{\frac{\mu}{\varepsilon}}^{\lambda} \frac{1}{f_\partial(v)}\, dv. \tag{3.5.6}$$

Now let $f_{\partial_i} \in C^1([\frac{\mu}{\varepsilon}, \lambda])$ $i = 1,2$ be any two distinct solutions of (3.3.5) satisfying (3.5.4) and (3.5.5) where ∂_1 and ∂_2 are constants with $0 < \partial_1 < \partial_2$.

It then follows from (3.5.5) and the uniqueness of solutions to the initial value problem for (3.3.5) that

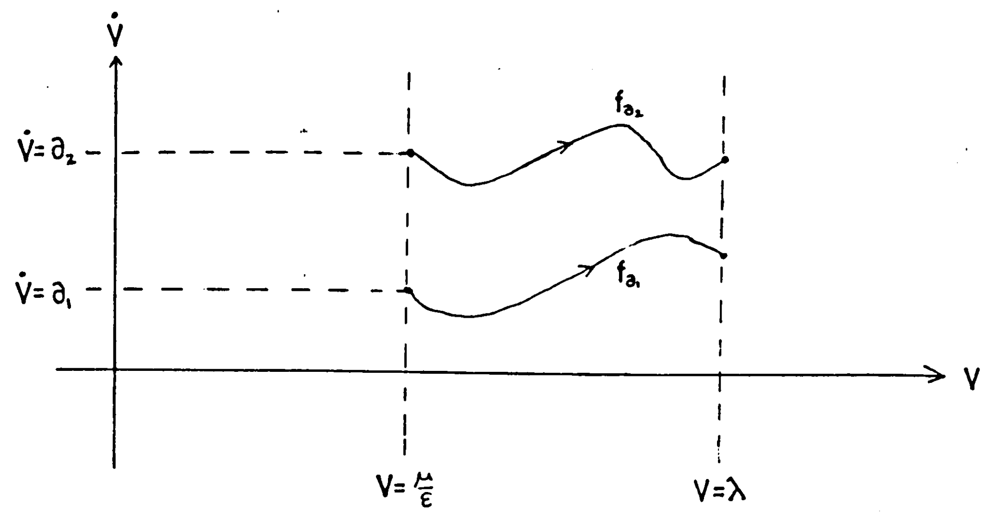
$$f_{\partial_1}(v) < f_{\partial_2}(v) \quad \text{for } v \in \left[\frac{\mu}{\varepsilon},\lambda\right] .$$

Hence

$$T^*(\partial_2) = \int_{\frac{\mu}{\varepsilon}}^{\lambda} \frac{1}{f_{\partial_2}(v)}\, dv < \int_{\frac{\mu}{\varepsilon}}^{\lambda} \frac{1}{f_{\partial_1}(v)}\, dv = T^*(\partial_1)$$

and so

$$T^*(\partial_1) > T^*(\partial_2). \tag{3.5.7}$$



<u>Step 3</u>

Now let $v_0 \in C^2([\log\varepsilon,0])$ be a non constant solution of (3.0.2) that satisfies (3.5.1)-(3.5.3). The arguments contained in remark 3.3 then imply that $v_0(s)$ generates a solution $f_{\partial_0} \in C^1([\frac{\mu}{\varepsilon},\lambda])$ of (3.3.5) satisfying (3.5.4) and (3.5.5) with $\partial_0 = \dot{v}_0(\log\varepsilon)$. It then follows that

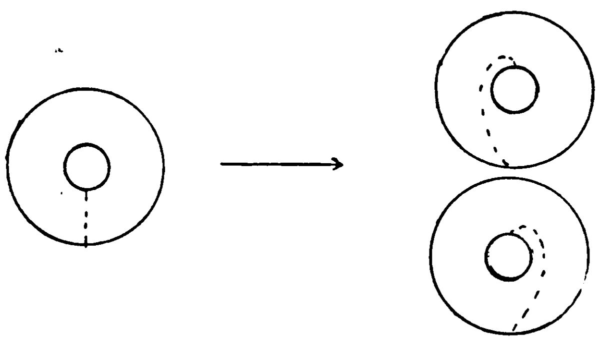
$$T^*(\partial_0) = \int_{\frac{\mu}{\varepsilon}}^{\lambda} \frac{1}{f_{\partial_0}(v)}\, dv = \int_{\log\varepsilon}^{0} \frac{1}{f_{\partial_0}(v_0(s))}\, ds = \int_{\log\varepsilon}^{0} 1 ds = \log\frac{1}{\varepsilon} . \tag{3.5.9}$$

The proof of the theorem is completed on noting that any two distinct solutions $v_i(s)$ $i=1,2$ of (3.0.2) that satisfy (3.5.1) - (3.5.3) will generate two distinct functions f_{∂_i} $i = 1,2$ satisfying the conditions (3.5.4) and (3.5.5) of Step 2; (3.5.7) and (3.5.9) then yield a contradiction.

Remark 3.6

The results of theorems 3.4 and 3.5 are uniqueness results purely within the class of radial deformations. For the displacement boundary value problem for a punctured ball we do not in general expect uniqueness of equilibrium solutions as the following example due to Fritz John shows.

Consider the displacement boundary value problem for a punctured ball in which the outer boundary is kept fixed and the inner one is subjected to a rotation of about a fixed axis. There exist at least two equilibrium solutions to this problem, one differing from the other by the sense in which the inner boundary is rotated (see diagram).



The non uniqueness arises from the presence of a disconnected boundary. For examples of local uniqueness results for the displacement boundary value problem we refer to Valent (1978) and for global results for star-shaped domains to Knops and Stuart (1984).

2. Uniqueness of Cavitating Solutions

In this section we make a change of variables in the energy functional relative to which the energy becomes a convex function (see proposition 3.13), which leads to a proof of uniqueness of cavitating equilibrium solutions in theorem 3.14.

First we state a proposition concerning the invertibility of the relation $v = \frac{r(R)}{R}$ when r is a cavitating equilibrium solution.

Proposition 3.7

Let $r \in C^2((0,1])$ be a cavitating equilibrium solution with $r(1) = \lambda$ for $\lambda > 0$. Then there exists a function $g:[\lambda,\infty) \longrightarrow (0,1]$, $g \in C^2([\lambda,\infty))$ satisfying

(i) $g(\frac{r(R)}{R}) = R^3$ for $R \in (0,1]$,

(ii) $g(\lambda) = 1$,

(iii) $\lim\limits_{V \to \infty} g(v) = 0$,

(iv) $$\frac{3g(\frac{r(R)}{R})}{g'(\frac{r(R)}{R})} + \frac{r(R)}{R} = r'(R) \quad \text{for } R \in (0,1] \ .$$

Proof

The existence of g satisfying (i) is a consequence of corollary 0.5 and the inverse function theorem. Conditions (ii) and (iii) then follow from (i) as does (iv) on implicit differentiation.

Proposition 3.8

Let Φ satisfy H1-H3. If $r \in C^2((0,1])$ is a cavitating equilibrium solution with $r(1) = \lambda > 0$, then

(i) $$\lim_{v \to \infty} \frac{1}{v^3}\left[\Phi(\frac{3g(v)}{g'(v)} + v,v,v) - \frac{3g(v)}{g'(v)}\,\Phi_{,1}(\frac{3g(v)}{g'(v)} + v,v,v)\right] = 0$$

(ii) $I(r) = \Phi(r'(1),\lambda,\lambda) + (\lambda - r'(1))\Phi_{,1}(r'(1),\lambda,\lambda) = H(\lambda,r'(1))$,

where g is defined as in proposition 3.7 and H is given by (0.7.2).

Proof

Condition (ii) is a direct consequence of proposition 0.16. From the proof of proposition 0.16 it also follows that

$$\lim_{R \to 0} R^3\left[\Phi(r'(R), \frac{r(R)}{R}, \frac{r(R)}{R}) + (\frac{r(R)}{R} - r'(R))\Phi_{,1}(r'(R), \frac{r(R)}{R}, \frac{r(R)}{R})\right] = 0.$$

Part (i) then follows from proposition 3.7 on setting $v = \frac{r(R)}{R}$, since $r(0) > 0$.

Remark 3.9

If $r \in C^2((0,1])$ is a cavitating equilibrium solution with $r(1) = \lambda$ and if g is defined as in proposition 3.7, then

$$g(v) = r^3(g^{\frac{1}{3}}(v)) \frac{1}{v^3} \quad \text{for } v \in [\lambda, \infty) \tag{3.9.1}$$

and hence

$$\frac{r^3(0)}{v^3} \leq g(v) \leq \frac{\lambda^3}{v^3} \quad \text{for } v \in [\lambda, \infty). \tag{3.9.2}$$

Remark 3.10

The function $H(X,Y)$ as defined by (0.7.2) satisfies

$$\frac{\partial}{\partial X} H(X,Y) > 0 \quad \text{for } X \in (0,Y).$$

whenever Φ satisfies H1.

Proposition 3.11

Suppose that Φ satisfies H1-H3 and that $r \in C^2((0,1])$ is a cavitating equilibrium solution with $r(1) = \lambda > 0$. Then the energy of the deformation is finite and given by

$$I(r) = I(g) \overset{\text{def}}{=} -\int_{\lambda}^{\infty} \frac{g'(v)}{3} \Phi\left(\frac{3g(v)}{g'(v)} + v, v, v\right) dv, \tag{3.11.1}$$

where g is defined as in proposition 3.7.

Proof

The energy $I(r)$ is finite by proposition 0.16 ; (3.11.1) then follows immediately from proposition 3.7 on noting that

$$g'(v) \frac{dv}{dR} = 3R^2.$$

We next show that the function g of proposition 3.7 satisfies the Euler-Lagrange equations corresponding to the functional $\tilde{I}$ as defined by (3.11.1).

Proposition 3.12

Let $r \in C^2((0,1])$ be a cavitating equilibrium solution. Then

$$\frac{d}{dv}\left[\frac{-1}{3}\,\Phi\left(\frac{3g(v)}{g'(v)}+v,v,v\right)+\frac{g(v)}{g'(v)}\,\Phi_{,1}\left(\frac{3g(v)}{g'(v)}+v,v,v\right)\right]=-\Phi_{,1}\left(\frac{3g(v)}{g'(v)}+v,v,v\right)$$

for $v \in [\lambda,\infty)$, (3.12.1)

where g is defined as in proposition 3.7.

Proof

As $g \in C^2([\lambda,\infty))$, (3.12.1) is equivalent to

$$\frac{-1}{3}\left[3-\frac{3gg''}{(g')^2}+1\right]\Phi_{,1}-\frac{2}{3}\,\Phi_{,2}+\left[1-\frac{gg''}{(g')^2}\right]\Phi_{,1}+\frac{g}{g'}\frac{d}{dv}\,\Phi_{,1}=-\Phi_{,1}$$

for $v \in [\lambda,\infty)$. (3.12.2)

This may be rewritten as

$$\frac{g}{g'}\frac{d}{dv}\left[\Phi_{,1}\left(\frac{3g}{g'}+v,v,v\right)\right]=\frac{2}{3}\left[\Phi_{,2}\left(\frac{3g}{g'}+v,v,v\right)-\Phi_{,1}\left(\frac{3g}{g'}+v,v,v\right)\right]. \quad (3.12.3)$$

The function r is a solution of (0.2.3) and hence

$$R\,\frac{d}{dR}\left[\Phi_{,1}\left(r',\frac{r}{R},\frac{r}{R}\right)\right]=2\left[\Phi_{,2}\left(r',\frac{r}{R},\frac{r}{R}\right)-\Phi_{,1}\left(r',\frac{r}{R},\frac{r}{R}\right)\right] \quad (3.12.4)$$

for $R \in (0,1]$.

On setting $v=\frac{r}{R}$, (3.12.4) takes the form (3.12.3) by proposition 3.7, completing the proof.

The last proposition is an example of the general invariant nature of the Euler-Lagrange equations (see Cesari (1983) p.48). We next examine a property of the integrand of $\tilde{I}$ as defined by (3.11.1).

Proposition 3.13

If Φ satisfies H1 then the function $G : S \longrightarrow \mathbb{R}$ defined by

$$G\binom{X_1}{X_2}=G(\underline{X})=\frac{-X_2}{3}\,\Phi_{,1}\left(\frac{3X_1}{X_2}+v,v,v\right) \quad (3.13.1)$$

is a convex function on S for each $v \in (0,\infty)$, where

$$S = \left\{ \underline{X} = \binom{X_1}{X_2} \in \mathbb{R}^2 \; ; \; X_1 \in (0,\infty),\; X_2 \in (-\infty,0),\; \frac{3X_1}{v} + X_2 < 0 \right\}. \tag{3.13.2}$$

Proof

Since S is a convex subset of $\mathbb{R}^2$ for each $v \in (0,\infty)$ it is sufficient to show that the hessian of G is positive semi-definite on S.
An easy calculation gives

$$\text{Hess } G(\underline{X}) \stackrel{\text{def}}{=} \left[\frac{\partial^2 G(\underline{X})}{\partial X_i X_j}\right] = 3\begin{bmatrix} \frac{-1}{X_2} & \frac{X_2}{X_2^2} \\ \frac{X_1}{X_2^2} & \frac{-X_1^2}{X_2^3} \end{bmatrix} \cdot \Phi_{,11}\left(\frac{3X_1}{X_2} + v, v, v\right). \tag{3.13.3}$$

It follows from H1, (3.13.3) and (3.13.2) that the trace of Hess $G(\underline{X})$ and the determinant of Hess $G(\underline{X})$ satisfy

$$\det(\text{Hess } G(\underline{X})) \equiv 0 \text{ and } \operatorname{tr}(\text{Hess } G(\underline{X})) > 0 \quad \text{for } \underline{X} \in S,$$

for each $v \in (0,\infty)$. Hence $G(\underline{X})$ is positive semidefinite, completing the proof.

We next state the main result of this section.

Theorem 3.14

If Φ satisfies H1-H3 then for each $\lambda \in [1,\infty)$ there exists at most one cavitating equilibrium $r \in C^2((0,1])$ satisfying $r(1) = \lambda$.

Proof

We suppose for a contradiction that there exists $\lambda \in [1,\infty)$ for which there are two distinct cavitating equilibrium solutions $r_i(R) \in C^2((0,1])$ with $r_i(1) = \lambda$ $i = 1,2$. Let g_i $i = 1,2$ be the corresponding functions as defined in proposition 3.7; then by proposition 3.11

$$I(r_i) = \tilde{I}(g_i) = -\int_\lambda^\infty g_i(v)\, \Phi\left(\frac{3g_i(v)}{g_i'(v)} + v, v, v\right) dv \quad i = 1,2\,. \tag{3.14.1}$$

It follows from proposition 3.13 that

$$\int_\lambda^M G\begin{bmatrix} g_1(v) \\ g_1'(v)\end{bmatrix} dv \leq \int_\lambda^M G\begin{bmatrix} g_2(v) \\ g_2'(v)\end{bmatrix} dv + \int_\lambda^M \frac{\partial G}{\partial X_1}\begin{bmatrix} g_1(v) \\ g_1'(v)\end{bmatrix}(g_1(v)-g_2(v)) +$$

$$\frac{\partial G}{\partial X_2}\begin{bmatrix} g_1(v) \\ g_2(v)\end{bmatrix}(g_1'(v)-g_2'(v))\, dv \quad \text{for each} \quad M\in(\lambda,\infty), \tag{3.14.2}$$

where G is defined by (3.13.1) (this is an elementary consequence of the convexity of G). Integrating the second integral on the right hand side of (3.14.2) by parts we obtain

$$\int_\lambda^M \frac{\partial G}{\partial X_1}\begin{bmatrix} g_1 \\ g_1'\end{bmatrix}(g_1-g_2) + \frac{\partial G}{\partial X_2}\begin{bmatrix} g_1 \\ g_1'\end{bmatrix}(g_1'-g_2')dv = \int_\lambda^M \left(\frac{\partial G}{\partial X_1}\begin{bmatrix} g_1 \\ g_1'\end{bmatrix} - \frac{d}{dv}\left[\frac{\partial G}{\partial X_2}\begin{bmatrix} g_1 \\ g_1'\end{bmatrix}\right]\right)(g_1-g_2)dv$$

$$+ \Bigg|_\lambda^M \left[(g_1-g_2)\,\frac{\partial G}{\partial X_2}\begin{bmatrix} g_1 \\ g_1'\end{bmatrix}\right]. \tag{3.14.3}$$

Proposition 3.12 and (3.13.1) then imply that the integrand on the right hand side of (3.14.3) is identically equal to zero. We thus obtain

$$\int_\lambda^M \frac{\partial G}{\partial X_1}\begin{bmatrix} g_1 \\ g_1'\end{bmatrix}(g_1-g_2) + \frac{\partial G}{\partial X_2}\begin{bmatrix} g_1 \\ g_1'\end{bmatrix}(g_1'-g_2')dv = \left[g_2(M)-g_1(M)\right]\frac{\partial G}{\partial X_2}\begin{bmatrix} g_1(M) \\ g_1'(M)\end{bmatrix} \tag{3.14.4}$$

for each $M\in(\lambda,\infty)$, where we have used the fact that $g_1(\lambda) = g_2(\lambda) = 1$. Remark 3.9, proposition 3.8(i) and (3.13.1) then imply that the right hand side of (3.14.4) tends to zero as M tends to infinity, thus (3.14.1) and (3.14.2) imply that

$$I(r_1) = \tilde{I}(g_1) \leq \tilde{I}(g_2) = I(r_2).$$

Interchanging the roles of r_1 and r_2 in the above arguments we obtain

$$H(\lambda, r_1'(1)) = I(r_1) = I(r_2) = H(\lambda, r_2'(1)),$$

where we have used proposition 3.8(ii). Corollary 0.5 implies that

$r_i'(1) < \lambda \quad i = 1,2$, and thus it follows from remark 3.10 that

$r_1'(1) = r_2'(1)$. Hence $r_1(R) \equiv r_2(R)$, a contradiction.

Remark 3.15

The theorem holds with H3 and H2 replaced by H7 (this follows analogously on using remark 0.16).

Finally in this chapter we indicate the proof of theorem 0.14. The uniqueness of r_c follows from theorem 3.14. Proposition 0.9 then implies that r_c is uniquely extendable to $r_c \in C^2((0,\infty))$ as a solution of (0.2.3) with $\frac{r_c(R)}{R} \searrow \lambda_c$ as $R \longrightarrow \infty$ and $\lambda_c \in (1,\lambda)$ by remark 0.10. It follows from proposition 1.1 that there exists a global minimiser $\tilde{r}$ of I on A_λ and proposition 0.3 implies that $\tilde{r}$ is a solution of (0.2.3). By proposition 1.6 $\tilde{r}(0) = 0$ if and only if $\tilde{r}(R) \equiv \lambda R$, and using proposition 0.16 and H1 we obtain

$$I(r_c) = \frac{1}{3}\left[\Phi(r_c'(1),\lambda,\lambda) + (\lambda - r_c'(1))\Phi_{,1}(r_c'(1),\lambda,\lambda)\right] < \frac{\Phi(\lambda,\lambda,\lambda)}{3} = I(\lambda R).$$

Thus $\tilde{r}(R) \not\equiv \lambda R$ and so $\tilde{r}(0) > 0$, hence $\tilde{r}$ is a cavitating equilibrium solution and so $\tilde{r}(R) \equiv r_c(R)$ by part (i). Finally, if $\mu \in (\lambda_c,\infty)$ and d is the unique root of $dr_c(\frac{1}{d}) = \mu$ then $r_{\tilde{c}}(R) \equiv dr_c(\frac{R}{d})$ is a cavitating equilibrium solution satisfying $r_{\tilde{c}}(1) = \mu$. The above arguments then imply that $r_{\tilde{c}}$ is the global minimiser of I on A_μ.

Corollary 0.15 follows on using remark 0.17 in place of proposition 0.16 in the above arguments.

CHAPTER 4

In section 1 of this chapter we present results on the asymptotic behaviour of solutions to the mixed problem for punctured balls studied in chapter 3. Our main result will be the determination of the first term in a uniform expansion for such solutions, this is given in theorem 4.9.

In section 2 we indicate possible applications of our results to problems concerning the interactions of holes in a material.

Proposition 4.1

Suppose Φ satisfies H1, E1 and for each $\lambda,\varepsilon > 0$ let

$$I_\varepsilon(r) \stackrel{\text{def}}{=} \int_\varepsilon^1 R^2\Phi(r'(R), \frac{r(R)}{R}, \frac{r(R)}{R})dR \tag{4.1.1}$$

whenever $r \in A_\lambda^\varepsilon$ where

$$A_\lambda^\varepsilon \stackrel{\text{def}}{=} \left\{r \in W^{1,1}(\varepsilon,1); r(1) = \lambda,\ r' > 0 \text{ a.e., } r(\varepsilon) \geqslant 0\right\}. \tag{4.1.2}$$

If $\{\varepsilon_n\}$ is a strictly positive sequence with $\varepsilon_n \longrightarrow 0$ as $n \longrightarrow \infty$ and if r_{ε_n} is a minimiser of I_{ε_n} on $A_\lambda^{\varepsilon_n}$, then there exist $r \in A_\lambda$ and a subsequence $\{\varepsilon_{n(j)}\}$ such that

$$r_{\varepsilon_{n(j)}} \xrightarrow{W^{1,1}(\partial,1)} r \quad \text{as} \quad j \longrightarrow \infty \tag{4.1.3}$$

for each $\partial \in (0,1)$.

Moreover

$$I(r) = \operatorname*{Inf}_{A_\lambda} I. \tag{4.1.4}$$

Proof

The existence of r_{ε_n} follows from analogous arguments to those used in proposition 1.1. Fixing $\partial \in (0,1)$ there exists $N(\partial)$ such that $0 < \varepsilon_n < \partial$ whenever $n > N(\partial)$. It then follows from E1 that

$$\int_0^1 \partial^2 \psi(r'_{\varepsilon_n})dR \leq I_\partial(r_{\varepsilon_n}) \leq I_{\varepsilon_n}(r_{\varepsilon_n}) = \inf_{A_\lambda^{\varepsilon_n}} I_{\varepsilon_n} \leq I_{\varepsilon_n}(\tilde{r}) \leq I(\tilde{r}) \quad \text{const.}$$

$$\text{for} \quad n > N(\partial), \tag{4.1.5}$$

where $\tilde{r}$ is any global minimiser of I on A_λ (there exists at least one by proposition 1.1). The De la Vallee Poussin theorem (c.f. Cesari (1983)) then implies the existence of a subsequence $\{r^\partial_{\varepsilon_n}\}$ of $\{r_{\varepsilon_n}\}$ which is weakly convergent in $W^{1,1}(\partial,1)$. Using the techniques of proposition 1.1 and choosing inductive subsequences $\left\{r^{\partial_k}_n\right\}$ of $\left\{r^{\partial_{k-1}}_n\right\}$ for some positive sequence $\{\partial_k\} \longrightarrow 0$ as $k \longrightarrow \infty$, it can be shown that the diagonal sequence then satisfies (4.1.3) for some $r \in A_\lambda$.

Finally, to prove (4.1.4) we note that for each $\partial \in (0,1)$ $r_{\varepsilon_{n(j)}} \in A^\partial_\lambda$ for j sufficiently large, hence (4.1.5) implies that

$$I_\partial(r_{\varepsilon_{n(j)}}) \leq I(\tilde{r}) \quad \text{for} \quad j \quad \text{sufficiently large.}$$

The weak lower semicontinuity of I_∂ then implies

$$I_\partial(r) \leq I(\tilde{r}).$$

But this holds for each $\partial \in (0,1)$ and so by the monotone convergence theorem

$$I(r) \leq I(\tilde{r}) \tag{4.1.6}$$

Since $r \in A_\lambda$ equality holds in (4.1.6).

We next prove a similar convergence result for the traction problem.

<u>Proposition 4.2</u>

Let Φ satisfy H1, E1 and let r_ε be a minimiser of I^ε_p on $\mathcal{B}^\varepsilon$ where

$$I^\varepsilon_p(r) = \int_\varepsilon^1 R^2\Phi(r'(R), \frac{r(R)}{R}, \frac{r(R)}{R})dR - Pr(1) \tag{4.2.1}$$

and

$$\mathcal{B}^{\varepsilon} = \left\{ r \in W^{1,1}(\varepsilon,1) \ ; \ r(\varepsilon) \geq 0, \ r' > 0 \ \text{ a.e.} \right\}. \tag{4.2.2}$$

Then for each positive sequence $\{\varepsilon_n\}$ with $\varepsilon_n \longrightarrow 0$ as $n \longrightarrow \infty$ there exist a subsequence $\{\varepsilon_{n(j)}\}$ and $r \in \mathcal{B}^{\varepsilon}$ such that

$$r_{\varepsilon_{n(j)}} \xrightarrow{W^{1,1}(\partial,1)} r \quad \text{as} \quad j \longrightarrow \infty \tag{4.2.3}$$

for each $\partial \in (0,1)$.

Moreover

$$I_p(r) = \operatorname*{Inf}_{\mathcal{B}} I. \tag{4.2.4}$$

Proof

Again the existence of the relevant minimisers is a consequence of the arguments of proposition 1.1. Fixing $\partial \in (0,1)$, there exists $N(\partial)$ such that $\varepsilon_n < \partial$ whenever $n > N(\partial)$, hence

$$I_{\partial}(r_{\varepsilon_n}) - Pr_{\varepsilon_n}(1) \leq \operatorname*{Inf}_{\mathcal{B}^{\varepsilon_n}} I_p^{\varepsilon_n} = I_p^{\varepsilon_n}(r_{\varepsilon_n}) \leq I_p^{\varepsilon_n}(\tilde{r}) \leq I_p(\tilde{r}) \leq \text{const.} \tag{4.2.5}$$

for $n > N(\partial)$,

where $\tilde{r}$ is any global minimiser of I_p on $\mathcal{B}$. Using E1 and the arguments of proposition 1.2 it follows that

$$2K_1 \int_{\partial}^{1} \frac{r_{\varepsilon_n}}{R} R^2 dR + K_2 \int_{\partial}^{1} R^2 r'_{\varepsilon_n} dR + K_3 + \frac{1}{2}\int_{\partial}^{1} R^2 \psi(r'_{\varepsilon_n}) dR - Pr_{\varepsilon_n}(1) \leq I\,(r_{\varepsilon_n}) - Pr_{\varepsilon_n}(1), \tag{4.2.6}$$

where K_1 and K_2 are positive and may be chosen to be arbitrarily large. On integrating the second term on the left hand side of (4.2.6) by parts we obtain

$$K_1 r_{\varepsilon_n}(\partial) \left[\frac{1-\partial^2}{2}\right] + K_1 \int_{\partial}^{1} R r_{\varepsilon_n} dR + K_2 \left[r_{\varepsilon_n}(1) - r_{\varepsilon_n}(\partial)\partial^2 - 2\int_{\partial}^{1} R r_{\varepsilon_n} dR \right]$$
$$+ \frac{1}{2}\int_{\partial}^{1} R^2 \psi(r'\ ) dR \leq I_{\partial}(r_{\varepsilon_n}) - Pr_{\varepsilon_n}(1).$$

On choosing $K_1 > 2K_2 > P$ and by (4.2.5) we obtain a uniform bound on $r_{\varepsilon_n}(1)$. The proof of the proposition is then completed in an exactly analogous manner to that of proposition 4.1.

Our next proposition concerns the properties of minimisers r_ε of I_ε and gives the existence of the punctured ball solutions whose uniqueness was proved in theorem 3.4.

Proposition 4.3

Suppose that Φ satisfies H1, H5, H11, E1, E2 and that $\lambda \in (1,\infty)$. Then for each $\varepsilon \in (0,1)$ there exists a minimiser r_ε of I_ε on A_λ^ε. Moreover $r_\varepsilon \in C^2([\varepsilon,1])$ is a solution of (0.2.3) and satisfies (3.0.4) - (3.0.7).

Proof

Applying the techniques of proposition 1.1 we obtain the existence of a minimiser r_ε of I_ε on A_λ^ε for each $\varepsilon \in (0,1)$. Identical arguments to those contained in the appendix then imply that $r_\varepsilon \in C^2((\varepsilon,1])$ and is a solution of (0.2.3) satisfying (3.0.4) and (3.0.5). It therefore suffices to show that r_ε satisfies $r_\varepsilon(\varepsilon) > 0$ since this implies that r_ε satisfies (3.0.7) (on using analogous arguments to those in the appendix). We suppose for a contradiction that $r_{\varepsilon_0}(\varepsilon_0) = 0$ for some $\varepsilon_0 \in (0,1)$. Since $\lambda > 1$ by assumption $r_{\varepsilon_0}(R_0) = R_0$ for some $R_0 \in (\varepsilon_0,1)$. By the optimality of r_{ε_0} it then follows that $\tilde{r}$ defined by

$$\tilde{r}(R) = \begin{cases} R & \text{if } R \in [R_0,1] \\ r_{\varepsilon_0}(R) & \text{if } R \in [\varepsilon_0,R_0] \end{cases}$$

satisfies

$$I_\varepsilon(\tilde{r}(R)) \leqslant I_\varepsilon(R),$$

contradicting H11.

We remark that similar arguments can be used to show that for ε fixed the deformed cavity size is a monotone function of the boundary displacement.

Our next proposition gives more detailed information on the convergence of the energy minimisers r_ε of I_ε.

Proposition 4.4

Suppose that Φ satisfies H1-H4, E1, E2 and that for each $\varepsilon > 0$ r_ε is a global minimiser of I_ε on A^ε_λ. Then

(i) if $\lambda > \lambda_{crit}$; $\sup_{[\varepsilon,1]} |r_\varepsilon(R) - r_c(R)| \longrightarrow 0$ as $\varepsilon \longrightarrow 0$, (4.4.1)

(ii) if $\lambda \leqslant \lambda_{crit}$; $\sup_{[\varepsilon,1]} |r_\varepsilon(R) - \lambda R| \longrightarrow 0$ as $\varepsilon \longrightarrow 0$, (4.4.2)

where r_c is the cavitating equilibrium solution (if there is no cavitation we set $\lambda_{crit} = \infty$).

Proof

It follows from proposition 4.1 and theorem 0.14 that for each $\partial \in (0,1)$

$$r_\varepsilon \xrightarrow{W^{1,1}(\partial,1)} r_t \quad \text{as } \varepsilon \longrightarrow 0 \text{ if } \lambda \leqslant \lambda_{crit}, \tag{4.4.3}$$

where

$$r_t(R) \equiv \lambda R$$

and

$$r_\varepsilon \xrightarrow{W^{1,1}(\partial,1)} r_c \quad \text{as } \varepsilon \longrightarrow 0 \text{ if } \lambda > \lambda_{crit}. \tag{4.4.4}$$

We first treat the case $\lambda \leqslant \lambda_{crit}$ and we suppose for a contradiction that (4.4.2) does not hold. Then there exist $\varepsilon_0 > 0$ and positive sequences $\{\varepsilon_n\}$, $\{x_n\}$ with the properties

(a) $\varepsilon_n \longrightarrow 0$ as $n \longrightarrow \infty$,

(b) $x_n \in [\varepsilon_n, 1]$ for each n,

(c) $|r_{\varepsilon_n}(x_n) - \lambda x_n| \geqslant \varepsilon_0$ for all n.

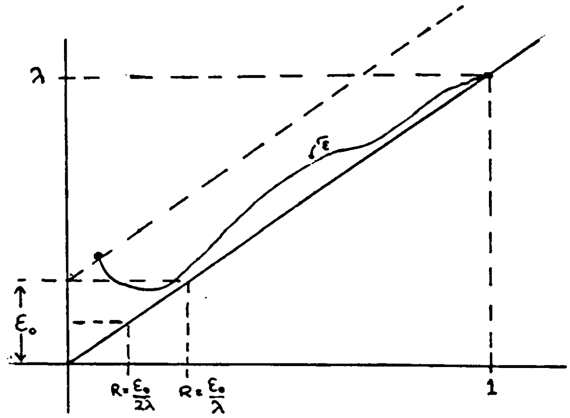
Condition (4.4.3) implies that for each $\partial \in (0,1)$

$$\sup_{[\partial,1]} \left| r_\varepsilon(R) - \lambda R \right| \longrightarrow 0 \quad \text{as} \quad \varepsilon \longrightarrow 0. \tag{4.4.5}$$

We may therefore assume without loss of generality that $x_n \longrightarrow 0$ as $n \longrightarrow \infty$,

and on choosing $\partial = \frac{\varepsilon_0}{2\lambda}$ we obtain a contradiction of the fact that

$r_\varepsilon'(R) > 0$ for $R \in (\varepsilon,1]$.



(r_ε would necessarily have the form indicated above).

We next consider the case $\lambda > \lambda_{crit}$. We again suppose for a contradiction that (4.4.4) does not hold. Then there exist $\varepsilon_0 > 0$ and positive sequences $\{\varepsilon_n\}$, $\{x_n\}$ with the properties

(a) $\varepsilon_n \longrightarrow 0$ as $n \longrightarrow \infty$, (4.4.6)

(b) $x_n \in [\varepsilon_n, 1]$ for each n, (4.4.7)

(c) $\left| r_{\varepsilon_n}(x_n) - r_c(x_n) \right| \geq \varepsilon_0$ for all n. (4.4.8)

Again (4.4.3) implies that for each $\partial \in (0,1)$

$$\sup_{[\partial,1]} \left| r\,(R) - r_c(R) \right| \longrightarrow 0 \quad \text{as}\, \varepsilon \longrightarrow 0 \tag{4.4.9}$$

and we therefore assume that

$$x_n \longrightarrow 0 \quad \text{as}\ n \longrightarrow \infty. \tag{4.4.10}$$

Note that $r_{\varepsilon_n} \in C^2([\varepsilon_n,1])$ for all n by proposition 4.3. We claim that

$$r'_{\varepsilon_n}(1) < r'_c(1) \quad \text{for all } n \ , \tag{4.4.11}$$

since if for some N $r'_c(1) < r'_{\varepsilon_N}(1)$ it then follows from H1 that

$$0 < T(r_c(1)) < T(r_{\varepsilon_N}(1)).$$

As $T(r_{\varepsilon_N}(\varepsilon)) = 0$ we conclude from proposition 0.6 that $r'_{\varepsilon_N}(R) < \frac{r_{\varepsilon_N}(R)}{R}$ for $R \in [\varepsilon_N,1]$. Consideration of the phase portrait together with H1 then implies that

$$0 \leqslant \frac{1}{v^2}\,\Phi,(r'_c(g^{\frac{1}{3}}(v)),v,v))\Bigg|_{v\,=\,\frac{r_{\varepsilon_N}(R)}{R}} < T(r_{\varepsilon_N}(R)) \tag{4.4.12}$$

for $R \in [\varepsilon_N,1]$,

where g is defined in proposition 3.7, contradicting the fact that r_{ε_N} satisfies (3.0.7); thus (4.4.11) holds. The continuity of r_c implies the existence of ∂_o such that

$$r_c(0) \leqslant r_c(R) \leqslant r_c(0) + \frac{\varepsilon_0}{3} \quad \text{for} \quad R \in (0,\partial_o] \ . \tag{4.4.13}$$

Setting $\partial = \partial_o$ in (4.4.9) we obtain

$$\left| r_{\varepsilon_n}(R) - r_c(R) \right| < \frac{\varepsilon_0}{3} \quad \text{for} \quad R \in [\partial_o,1] \ , \tag{4.4.14}$$

for sufficiently large n. Hence

$$r_{\varepsilon_n}(\partial_o) \leqslant r_c(\partial_o) + \frac{\varepsilon_0}{3} \leqslant r_c(0) + \frac{2\varepsilon_0}{3} \tag{4.4.15}$$

if n is sufficiently large, where we have used (4.4.13). By the arguments of theorem 3.5

$$r_c(R) \neq r_{\varepsilon_n}(R) \quad \text{for} \quad R \in [\varepsilon_n,1),$$

and so by (4.4.11) and (4.4.8) we conclude that

$$r_{\varepsilon_n}(x_n) \geq r_c(x_n) + \varepsilon_o \geq r_c(0) + \varepsilon_o \qquad (4.4.16)$$

for sufficiently large n by (4.4.13) and using (4.4.10). Conditions (4.4.16) and (4.4.15) together contradict (3.0.4) for large n.

Proposition 4.5

Let Φ satisfy H1, H2, H3, H4, H5, E1, E2. Then if ε is sufficiently small $r_\varepsilon \in C^2([\varepsilon,1])$ is a solution of (0.2.3) satisfying (3.0.4) - (3.0.7) if and only if it is the unique global minimiser of I_ε on A^ε_λ. Moreover

(i) if $\lambda \leq \lambda_{crit}$ then $\sup\limits_{[\varepsilon,1]} |r_\varepsilon(R)-\lambda R| \longrightarrow 0$ as $\varepsilon \longrightarrow 0$ and

(ii) if $\lambda > \lambda_{crit}$ then $\sup\limits_{[\varepsilon,1]} |r_\varepsilon(R)-r_c(R)| \longrightarrow 0$ as $\varepsilon \longrightarrow 0$.

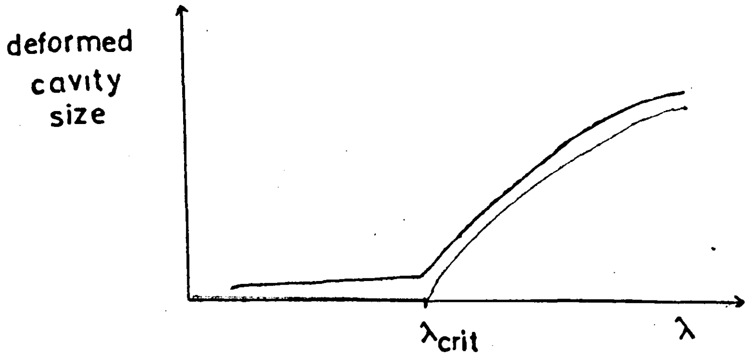
Proof

It follows from the arguments of proposition 4.3 and theorem 0.14 that a global minimiser r_ε always exists and satisfies (0.2.3) and (3.0.4) - (3.0.7). Theorem 3.4 implies that r_ε is unique; proving the first half of the proposition. The second half follows on applying proposition 4.4.

Remark 4.6

The above proposition holds with H3 and H4 replaced by H7; this is a consequence of corollary 0.15.

The preceding results then strongly suggest that the following type of bifurcation breaking occurs; as illustrated in the figure below where we have plotted the deformed cavity size against the corresponding boundary displacement λ. The red line represents the values of $r_\lambda(0)$ where r_λ is the minimiser of I on A_λ and the dark line represents the values of $r^\lambda_\varepsilon(\varepsilon)$ where r^λ_ε is the minimiser of I_ε on A^ε_λ (ε is fixed and chosen to be small).



We next examine the existence, uniqueness and asymptotic behaviour of solutions $r_o \in C^2([1,\infty))$ of (0.2.3) that satisfy

(i) $\frac{r_o(R)}{R} \longrightarrow \lambda$ as $R \longrightarrow \infty$ for some $\lambda \in (0,\infty)$, (4.6.1)

(ii) $r_o'(R) > 0$ for $R \in [1,\infty)$, (4.6.2)

(iii) $r_o(1) > 0$ and (4.6.3)

(iv) $T(r_o(1)) = 0$. (4.6.4)

We will use r_o to obtain the first term in an asymptotic expansion for solutions of (0.2.3) satisfying (3.0.4) - (3.0.7).

Proposition 4.7

If Φ satisfies E1, E2, H2$^+$, H3-H6, H8 then for each $\lambda \in (1,\lambda_{crit})$ there exists a unique solution $r_o \in C^2([1,\infty))$ satisfying (4.6.2) - (4.6.5) (where we set $\lambda_{crit} = \infty$ if there is no cavitation).

Proof

(i) We first consider the case where λ_{crit} is finite and we fix $\lambda \in (1,\lambda_{crit})$. As noted in remark 3.3 the integral curves of (3.3.5) are invariant under the flow generated by (3.0.2). Standard results for ordinary differential equations then imply that there exists a solution $f \in C^1((a,b))$ of (3.3.5) with $a < \lambda < b$ and such that $f(\lambda) = 0$. If $v(s)$ is a solution to the initial value problem for (3.0.2) with data $(v(0),\dot{v}(0)) = (v_o, f(v_o))$ where $v_o \in (\lambda, b)$; then $v(s)$ exists for $s \in [0,\infty)$. The solution $v(s)$ also exists for $s \in (-c,0]$, where $c > 0$ is chosen to be maximal. To prove the proposition it is sufficient to show that $(v(s),\dot{v}(s))$ crosses the zero stress curve for some $s \in (-c,\infty)$; i.e. that $\dot{v}(s) = \sigma(v(s))$ for some $s \in (-c,\infty)$ (where σ is as defined in proposition 3.1). Suppose for a contradiction that this does not occur, then we claim that $c = \infty$. If not $v(s)$ would induce a solution r of (0.2.3) on (e^{-c},∞) satisfying $\frac{r(R)}{R} \longrightarrow \lambda$ as $R \longrightarrow \infty$ and $\frac{r(R)}{R} \longrightarrow \infty$ as $R \longrightarrow e^{-c}$ contradicting the fact that $r'(R) > 0$ for $R \in (e^{-c},\infty)$.

The proof now proceeds by using an argument which in phase space corresponds to 'sandwiching' the solution curve $(v(s),\dot{v}(s))$ between the cavitating solution and the zero stress curve; it follows from consideration of the phase portrait and proposition 1.6 that $r(0) > 0$ and that for R sufficiently small

$$0 < T(r(R)) < \left(\frac{R}{r(R)}\right)^2 \Phi_{,1}\left(r_c'\left(g_c^{\frac{1}{3}}\left(\frac{r(R)}{R}\right)\right), \frac{r(R)}{R}, \frac{r(R)}{R}\right) \tag{4.7.1}$$

by H1, where r_c is the cavitating equilibrium solution and g_c is as defined in proposition 3.7. Finally as r_c satisfies $T((r_c(0))) = 0$ it follows from (4.7.1) that $\lim_{R \to 0} T(r(R)) = 0$ contradicting theorem 3.14.

(ii) We now consider the case when $\lambda_{crit} = \infty$ and we suppose for a contradiction that the proposition does not hold for some $\lambda_o \in (1,\infty)$. Then, using the arguments of (i) there exists a solution $r \in C^2((0,\infty))$ of (0.2.3) satisfying

(a) $0 < r'(R) < \frac{r(R)}{R}$ for $R \in (0,\infty)$,

(b) $\frac{r(R)}{R} \longrightarrow \lambda_o$ as $R \longrightarrow \infty$.

It then follows from (a) and proposition 0.6 that

$$T(r(R)) \longrightarrow C \quad 0 \text{ as } R \longrightarrow 0.$$

Hence for $\mu \in (\lambda_o,\infty)$ there exists a solution $\tilde{r} \in C^2((0,1])$ of (0.2.3) with

(c) $\tilde{r}(1) = \mu$,

(d) $0 < \tilde{r}'(R) < \frac{\tilde{r}(R)}{R}$ for $R \in (0,1]$,

(e) $T(\tilde{r}(R)) \searrow C$ as $R \longrightarrow 0$,

(we simply choose $\tilde{r}$ to be a suitable rescaling of r). Using analogous arguments to those of proposition 0.16 we conclude that $\tilde{r}$ has finite energy and that

$$I(\tilde{r}) = H(\mu,\tilde{r}'(1)) - \lim_{R \longrightarrow 0} R^3 H(\frac{\tilde{r}(R)}{R}, \tilde{r}'(R)), \tag{4.7.2}$$

where H is defined by (0.7.3). It follows from (d) and (e) that $R^3 H(\frac{\tilde{r}(R)}{R}, \tilde{r}'(R)) > 0$ for $R \in (0,1)$ and hence by (4.7.2) that

$$I(r) \leq H(\mu,r'(1)) < \frac{\Phi(\mu,\mu,\mu)}{3} = I(\mu R), \tag{4.7.3}$$

where we have used H1. This is a contradiction of propositions 1.1, 1.6 and 0.3 since by assumption there is no cavitation.

Proposition 4.8

If $r_o \in C^2([1,\infty))$ is a solution of (0.2.3) satisfying (4.6.2) and (4.6.3) then

$$r_o(R) = \lambda R + O(\frac{1}{R^2}). \tag{4.8.1}$$

Proof

Let $h \in C^2((a,b))$ be a solution of (3.3.5), where $\lambda \in (a,b)$ and h satisfies $h(\lambda) = 0$; then by Taylor's theorem

$$h(v) = -3(v-\lambda) + g(v-\lambda), \tag{4.8.2}$$

where

$$\left|\frac{g(v-\lambda)}{(v-\lambda)^2}\right| < \text{constant for } |v-\lambda| \text{ sufficiently small.} \tag{4.8.3}$$

Hence if $v(s)$ is the solution of (3.0.2) corresponding to r_o under the change of variables (3.0.1), then

$\dot{v}(s) = h(v(s)) = -3(v(s)-\lambda) + g(v(s)-\lambda)$ if s is sufficiently large.

As $v(s) > \lambda$ for all s, standard arguments imply that

$$e^{3s}\,|v(s)-\lambda| < \text{const. for all } s$$

and (4.8.1) holds by (3.0.1).

Theorem 4.9

Suppose that Φ satisfies E1, H1, H2$^+$, H3-H8 and that $r_\varepsilon \in C^2([\varepsilon,1])$ is a solution of (0.2.3) satisfying (3.0.4) - (3.0.7) with $\lambda \in (1,\lambda_{crit})$. Then

$$r_\varepsilon(R) = \varepsilon r_o\left(\tfrac{R}{\varepsilon}\right) + o(\varepsilon), \tag{4.9.1}$$

where $r_o \in C^2([1,\infty))$ is as defined in proposition 4.7.

Proof

We define r^ε by

$$r^\varepsilon(\eta) = \frac{1}{\varepsilon}\, r_\varepsilon(\varepsilon\eta) \quad \text{for} \quad \eta \in \left[1,\tfrac{1}{\varepsilon}\right], \tag{4.9.2}$$

It then follows that $r \in C^2([1, \frac{1}{\varepsilon}])$ is a solution of (0.2.3) for each $\varepsilon \in (0,1)$ and satisfies

$$\text{(i)} \quad r^\varepsilon(\tfrac{1}{\varepsilon}) = \frac{\lambda}{\varepsilon}\,, \tag{4.9.3}$$

$$\text{(ii)} \quad r^{\varepsilon\,\prime}(\eta) > 0 \quad \text{for} \quad \eta \in \left[1, \tfrac{1}{\varepsilon}\right], \tag{4.9.4}$$

$$\text{(iii)} \quad \Phi_{,1}(r^{\varepsilon\,\prime}(1), r^\varepsilon(1), r^\varepsilon(1)) = 0\,. \tag{4.9.5}$$

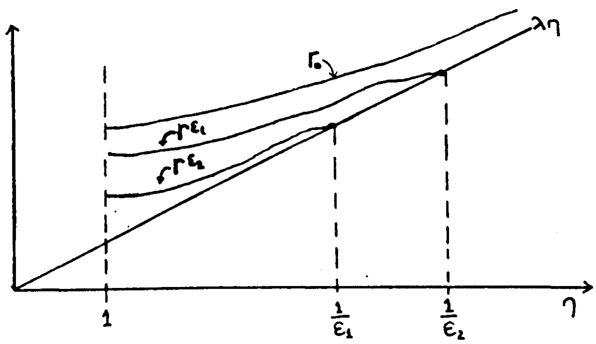
The r^ε exhibit the following monotonicity property; if $0 < \varepsilon_2 < \varepsilon_1$ then

$$r^{\varepsilon_1}(\eta) < r^{\varepsilon_2}(\eta) \quad \text{for} \quad \eta \in \left[1, \frac{1}{\varepsilon_1}\right]; \tag{4.9.6}$$

otherwise by (4.9.3) there would exist $\eta_o \in \left[1, \frac{1}{\varepsilon_1}\right]$ with

$r^{\varepsilon_1}(\eta_o) = r^{\varepsilon_2}(\eta_o) \overset{\text{def}}{=} \lambda_o\eta_o$ and the r^{ε_i} would correspond to two distinct solutions of the mixed displacement/traction problem on $[1,\eta_o]$ contradicting theorem 3.4. It follows by an analogous argument that

$$r_o(\eta) > r^\varepsilon(\eta) \quad \text{for} \quad \eta \in \left[1, \tfrac{1}{\varepsilon}\right], \ \varepsilon \in (0,1). \tag{4.9.7}$$



Moreover H8, (4.6.5), H1 and proposition 0.6 then allow us to conclude further that

$$r_o(\eta) \geq r^\varepsilon(\eta) \geq \lambda\eta \quad \text{for} \quad \eta \in \left[1, \tfrac{1}{\varepsilon}\right], \ \varepsilon \in (0,1). \tag{4.9.9}$$

Using (4.9.2) we obtain

$$\frac{1}{\varepsilon} \sup_{[\varepsilon,1]} \left| r_\varepsilon(R) - \varepsilon r_o(\tfrac{R}{\varepsilon}) \right| = \sup_{[1,\frac{1}{\varepsilon}]} \left| r^\varepsilon(\eta) - r_o(\eta) \right| \tag{4.9.10}$$

for each $\varepsilon \in (0,1)$.

Thus in order to prove (4.9.1) it is sufficient to show that

$$\sup_{[1,\frac{1}{\varepsilon}]} |r^{\varepsilon}(\eta)-r_o(\eta)| \longrightarrow 0 \quad \text{as } \varepsilon \longrightarrow 0. \tag{4.9.11}$$

Given $\partial > 0$ it follows from proposition 4.8 that there exists a constant M_1 such that

$$\sup_{[M_1,\frac{1}{\varepsilon}]} |r_o(\eta)-\lambda\eta| < \frac{\partial}{2} \quad \text{whenever } \varepsilon < \frac{1}{M_1}. \tag{4.9.12}$$

To prove that (4.9.11) holds it is therefore sufficient by (4.9.12) and (4.9.9) to show that

$$\sup_{[1,M]} |r_o(\eta)-r^{\varepsilon}(\eta)| \longrightarrow 0 \quad \text{as } \varepsilon \longrightarrow 0 \tag{4.9.13}$$

for each $M \in (1,\infty)$.

We suppose for a contradiction that this does not hold; then there exist $M_o \in (1,\infty)$, $\partial_o \in (0,\infty)$ and a sequence $\{\varepsilon_n\}$ with $\varepsilon_n \longrightarrow 0$ as $n \longrightarrow \infty$ such that

$$\sup_{[1,M_o]} |r_o(\eta)-r_{\varepsilon_n}(\eta)| \geq \partial_o \quad \text{for all } n. \tag{4.9.14}$$

It follows from proposition 0.8 that

$$\int_1^{M_o} |r^{\varepsilon_n''}(\eta)|\,d\eta = \int_1^{M_o} r^{\varepsilon_n''}(\eta)\,d\eta < 2r^{\varepsilon_n'}(M_o) < 2\lambda, \tag{4.9.15}$$

for all n.

Also

$$\int_1^{M_o} |r^{\varepsilon_n'}(\eta)|\,d\eta = \int_1^{M_o} r^{\varepsilon_n'}(\eta)\,d\eta \leq r^{\varepsilon_n}(M_o) \leq r_o(M_o) \tag{4.9.16}$$

for all n by (4.9.9).

Conditions (4.9.15) and (4.9.16) imply that $\{r^{\varepsilon_n}\}$ is a bounded sequence in $W^{2,1}(1,M_o)$ and since the embedding

$$W^{2,1}(1,M_o) \longrightarrow W^{1,1}(1,M_o)$$

is compact we may assume without loss of generality that

$$r^{\varepsilon_n} \longrightarrow \tilde{r} \quad \text{as} \quad n \longrightarrow \infty \text{ pointwise for a.e. } \eta \in (1,M_o) \text{ and} \tag{4.9.17}$$

$$r^{\varepsilon_n'} \longrightarrow \tilde{r}' \quad \text{as} \quad n \longrightarrow \infty \text{ pointwise for a.e. } \eta \in (1,M_o), \tag{4.9.18}$$

for some $\tilde{r} \in W^{1,1}(1,M_o)$ (where by assumption $r \neq r_o$).

On writing (0.2.3) in a weak form it follows that each r satisfies

$$\eta^2 \Phi_{,1}(r^{\varepsilon_n'}(\eta), \frac{r^{\varepsilon_n}(\eta)}{\eta}, \frac{r^{\varepsilon_n}(\eta)}{\eta}) = 2 \int_1^{\eta} s\Phi_{,2}(r^{\varepsilon_n'}(s), \frac{r^{\varepsilon_n}(s)}{s}, \frac{r^{\varepsilon_n}(s)}{s}) ds \tag{4.9.19}$$

for $\eta \in \left[1, \frac{1}{\varepsilon_n}\right]$, where we have incorporated the boundary condition (4.9.5).

From corollary 0.5 and (4.9.9) we conclude that

$$\lambda \leqslant \frac{r^{\varepsilon_n}(\eta)}{\eta} < r_o(1) \quad \text{for } \eta \in \left[1, \frac{1}{\varepsilon_n}\right], \quad \text{for all n.} \tag{4.9.20}$$

By proposition 0.8, (4.9.20), (4.9.5) and the continuity of $\Phi_{,1}$ there exists a constant $c > 0$ such that

$$c < r^{\varepsilon_n'}(1) < r^{\varepsilon_n'}(\eta) < \lambda \quad \text{for } \eta \in \left[1, \frac{1}{\varepsilon_n}\right], \text{ for all n.} \tag{4.9.21}$$

On using (4.9.17), (4.9.18), (4.9.20) and (4.9.21) we can pass to the limit in (4.9.19) by the dominated convergence theorem to conclude that $\tilde{r}$ satisfies (4.9.19) for a.e. $\eta \in [1,M_o]$. It then follows from theorem 0.2 that $\tilde{r}$ satisfies (0.2.3). By proposition 0.9 $\tilde{r}$ may be extended to $[1,\infty)$ as a solution of (0.2.3) and as

$$\tilde{r}(\eta) < r_o(\eta) \quad \text{for all } \eta \in [1,\infty)$$

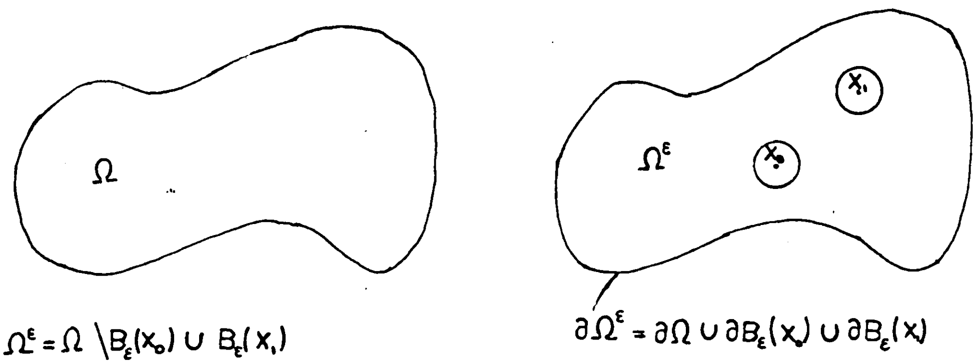
(which follows on using exactly analogous arguments to those used in obtaining (4.9.7)) we conclude from (4.6.2) that

$$\frac{\tilde{r}(\eta)}{\eta} \longrightarrow \lambda \quad \text{as} \quad \eta \longrightarrow \infty$$

Note also that $\tilde{r}$ satisfies (4.6.5) since it was a solution of (4.9.19). Hence r_0 and $\tilde{r}$ are distinct solutions of (0.2.3) satisfying (4.6.2) - (4.6.5), contradicting the uniqueness result of proposition 4.7.

2. Formal Expansions and Proving Validity

Our results for 'punctured' balls in chapter 3 section 1 and the first half of this chapter are of interest in studying the interactions between holes in a material; consider a mixed displacement/traction boundary value problem for a material with stored energy function W and having the following reference configuration Ω :



If W is a polyconvex stored energy function satisfying growth hypotheses that prevent cavitation and we look for energy minimising configurations $\underline{x}^\varepsilon(\underline{X})$ for Ω^ε satisfying

$$\underline{x}^\varepsilon(\underline{X}) = \lambda\underline{X} \quad \text{for} \quad \underline{X} \in \partial\Omega$$

and zero traction on the cavity surfaces, then using the existence theorems of Ball (1977) and the techniques of proposition 4.1 we can show that

$$\underline{x}^\varepsilon(\underline{X}) \longrightarrow \lambda\underline{X} \quad \text{uniformly as} \quad \varepsilon \longrightarrow 0.$$

For simplicity we now consider the problem of a <u>single</u> hole of radius centred at $\underline{X}_0$ in Ω . We will be interested in the first order effect of the hole on the surrounding material.

Formal Derivation of an Expansion

For small values of ε we expect equilibrium deformations to be radial to first order in the vicinity of the cavities. Guided by our results for a central hole in a ball of elastic material we expect boundary layer effects together with significant changes in strain in a neighbourhood of the holes. We therefore rescale variables in the boundary layer by setting

$\eta = \dfrac{\underline{X} - \underline{X}_0}{\varepsilon}$ and try the following leading terms in the inner and outer expansions

Outer: $\lambda\underline{X}$

Inner: $\dfrac{\varepsilon r_0(|\eta|)}{\eta}\eta$,

where r_0 is as in proposition 4.7. Applying the method of matched asymptotic expansions (see Fraenkel (1969) and Eckhaus (1979)) we obtain the following first order correction to the outer expansion

$$\text{Outer} : \lambda\underline{X} + \varepsilon^3\Big(\underline{f}(\underline{X}) + \frac{A(\underline{X}-\underline{X}_0)}{|\underline{X}-\underline{X}_0|^3}\Big),$$

where f is a solution in Ω of the equilibrium equations linearised around $\lambda\underline{X}$ with

$$\underline{f}(\underline{X}) = \frac{A(\underline{X}_0-\underline{X})}{|\underline{X}-\underline{X}_0|^3} \quad \text{for} \quad \underline{X} \in \partial\Omega$$

and where

$$r_0(R) = \lambda R + \frac{A}{R^2} + o\Big(\frac{1}{R^2}\Big).$$

Validity of the Formal Expansion

Generally, the first step towards a rigorous proof of the validity of such expansions is to combine the inner and outer expressions; one hopes that the composite expansion formed will be uniformly valid. This idea was formalised by Fraenkel (1969) together with the asymptotic matching principle in terms of inner and outer expansion operators (see also Eckhaus (1979)). Van Harten was able to prove the validity of formal expansions for certain singular perturbation problems through application of the contraction mapping principle in a suitably chosen Banach space (his particular contribution was in using norms that depended on the perturbation parameter); see Van Harten (1978), Eckhaus (1979). His proofs relied on careful use of the maximum principle (c.f. Protter and Weinberger (1967)) to obtain bounds on the solution of the equations linearised around the formal expansion.

In the case of radial deformations of a punctured ball and for a separable stored energy function it is possible to prove a stronger version of proposition 4.9; namely that

$$r_\varepsilon(R) = \varepsilon r_o(\tfrac{R}{\varepsilon}) + O(\varepsilon^3).$$

(The proof makes use of the contraction mapping principle and employs some of the techniques of Van Harten (1978)). However, for more complicated stored energy functions the method fails through inapplicability of the maximum principle, but a perturbation theorem due to Morrey (see Morrey (1966)) may be of relevance here and in the general three dimensional case.

Work on metals (e.g. Cox and Low (1974), Hancock and Cowling (1977)) indicates that void nucleation and coalescence is a possible mechanism for the initiation of fracture. This type of ductile fracture is often considered to be a phenomenon of plasticity. However, the successful use of Rice's J-integral in non linear fracture mechanics (see Rice (1968), Eshelby (1975)) indicates that this type of phenomenon could be treated within the framework of non linear elasticity provided unloading does not take place. We conjecture that in weak materials high stresses between the holes will give rise to cavitation between them. The cumulative effect of this across a body could be a mechanism for the initiation of fracture, with the creation of a line of holes leading to the formation of a crack.

In section 1 we present elements of the classical Weierstrass theory of the Calculus of variations. For ease of presentation we restrict attention to the problem of minimising

$$J(y) \overset{\text{def}}{=} \int_0^1 f(x,y(x),y'(x))\,dx \tag{5.0.1}$$

on

$$A \overset{\text{def}}{=} \left\{ y(x) \in W^{1,1}(0,1)\ ;\ y(0) > \alpha\ ,\ y(1) = \beta \right\}, \tag{5.0.2}$$

where f is a positive C^2 function and $\alpha, \beta > 0$ are constants. For a precise statement of the general theory and further details we refer to Cesari (1983) or to L. C. Young (1980), Morrey (1966), Bolza (1973).

In section 2 we use the theory developed to prove uniqueness of the cavitating equilibrium solution and to provide a general interpretation of cavitation via this field theory.

1. Classical Field Theory

Definition 1. We say that $y_o \in A$ is a strong local minimum of J on A if for some $\varepsilon > 0$

$$J(y_o) < J(y) \tag{5.0.3}$$

for all $y \in A$ with $\|y - y_o\|_\infty < \varepsilon$.

Definition 2. The Weierstrass excess function $\mathcal{E} : \mathbb{R}^4 \longrightarrow \mathbb{R}$ corresponding to the integrand f is given by

$$\mathcal{E}(x,y;p,q) \overset{\text{def}}{=} f(x,y,q) - f(x,y,p) + (p-q)\, f_p\,(x,y,p). \tag{5.0.4}$$

where f_p denotes differentiation of f with respect to its third argument. It is well known that if $y_o \in C^1([0,1]) \cap A$ is a strong local minimum of I on A then

$$\mathcal{E}(x,y_o(x);y_o'(x),q) > 0 \quad \text{for all } q \in \mathbb{R},\ x \in (0,1). \tag{5.0.5}$$

In higher dimensions the analogous conditions are those of quasiconvexity and strong ellipticity (c.f. Giaquinta (1983)) which imply (5.0.5) in the case of dimension one.

Definition 3. If $D \subset \mathbb{R}$ is connected, we say that $y \in C^2(D)$ is an extremal of J on D if y is a solution of

$$\frac{d}{dx} f_p(x,y(x),y'(x)) = f_u(x,y(x),y'(x)) \quad \text{for } x \in D, \tag{5.0.6}$$

where f_u denotes differentiation of f with respect to its second argument; (5.0.6) is the Euler-Lagrange equation for (5.0.1).

Definition 4. If $S \subset \mathbb{R}^2$ is an open simply connected region, we say that the one-parameter family of functions $\{y(x,\alpha); \alpha \in \tilde{A}\}$ with $\tilde{A} \subset \mathbb{R}$ constitutes a field of extremals $\mathcal{F}$ of J over S if

(i) for each $(a,b) \in S$ there exists a unique $\alpha_o \in \tilde{A}$ such that

$$y(a,\alpha_o) = b, \tag{5.0.7}$$

(ii) for each $\alpha \in \tilde{A}$ $y(x,\alpha) \in C^2(D_\alpha)$ is a solution of (5.0.6) where D_α is a subset of $\mathbb{R}$ satisfying

$$D_\alpha \supset \{x \in \mathbb{R} \; ; \; (x,y(x,\alpha)) \in S\}. \tag{5.0.8}$$

Definition 5. We define the slope function $P : \mathbb{R}^2 \longrightarrow \mathbb{R}$, $P \in C^1(S)$ corresponding to the field of extremals $\mathcal{F}$ by

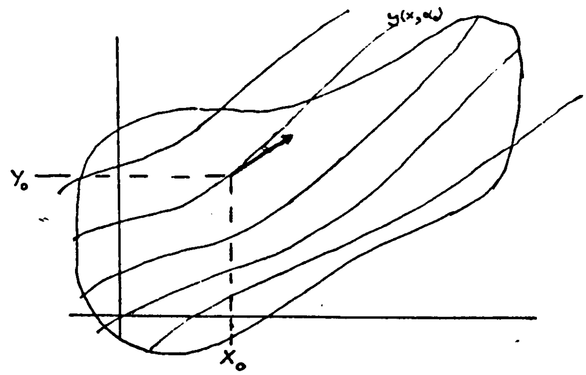
$$P(a,b) = \frac{d}{dx} y(x,\alpha_o) \Big|_{x=a} \quad \text{for } (a,b) \in S, \tag{5.0.9}$$

where $\alpha_o \in \tilde{A}$ is the unique element satisfying $y(a,\alpha_o) = b$.

i.e. $P(a,b)$ is the slope at (a,b) corresponding to the unique extremal of the field passing through the point (a,b) (see figure 1)

Figure 1

($P(x_o,y_o)$ is the slope of the arrow indicated)



Definition 6. We define the Hilbert integral J^* relative to f and the field $\mathcal{F}$ by

$$J^*(z) = \int_0^1 f(x,z(x),P(x,z(x))) + (z'(x)-P(x,z(x)))f_p(x,z(x),P(x,z(x)))\,dx$$

for $z \in A$, (5.0.10)

where P is as in definition 5. (The integral (5.0.10) clearly exists by our differentiability assumptions on f, p and z).

Definition 7. For each $y \in C([0,1])$ the graph of y $Gr(y)$ is given by

$$Gr(y) = \{ (x,y(x)) \; ; \; x \in [0,1] \} . \tag{5.0.11}$$

Definition 8. We say that $y_0 \in C^2([0,1])$ is imbedded in the field of extremals $\mathcal{F}$ if

(i) $Gr(y_0) \subset S$, (5.0.12)

(ii) $y_0(x) = y(x,\alpha^*)$ for all $x \in [0,1]$, (5.0.13)

for some $\alpha^* \in \tilde{A}$.

The following result is well known (c.f. Cesari (1983) for the proof).

Proposition 5.1

Let $\mathcal{F}$ be a field of extremals of I over a region $S \subset \mathbb{R}^2$ which is open and simply connected. Then there exists a function $H^* : \mathbb{R}^3 \longrightarrow \mathbb{R}$ with the property that

(i) $H^*(x,z(x),P(x,z(x))) \in W^{1,1}(0,1)$ (5.1.1)

and

(ii) $J^*(z) = H^*(x,z(x),P(x,z(x))) \Big|_0^1$, (5.1.2)

where (i) and (ii) hold for all $z \in W^{1,1}(0,1)$ with $Gr(z) \subset S$.

Using this proposition we can prove the next result which is a modified version of a theorem found in Cesari (1983) p.73.

Theorem 5.2

Suppose that the extremal $y_o \in C^2([0,1])$ with $y_o \in A$ is embedded in a field of extremals $\mathcal{F}$ over S (an open simply connected subset of $\mathbb{R}^2$) and that $Gr(y_o) \subset S$. Let

$$\mathcal{E}(x,y\ ;\ P(x,y),q) > 0 \quad \text{for all } (x,y) \in S, \text{ for all } q \in \mathbb{R} \quad q \neq P(x,y) \qquad (5.2.1)$$

where P is the slope function corresponding to $\mathcal{F}$. Then

$$J(y_o) < J(y)$$

for all $y \in A \quad y \not\equiv y_o$ satisfying $Gr(y) \subset S$ and such that

$$H^*(s,y(x),P(x,y(x))) = H^*(x,y_o(x),P(x,y_o(x))) \qquad (5.2.2)$$

at $x = 0,1$.

Proof

It follows from (5.2.1) and (5.0.4) that

$$J(y) > J^*(y) = H^*(x,y(x),y'(x))\Big|_0^1 = H^*(x,y_o(x),y_o'(x))\Big|_0^1 = J^*(y_o) = J(y_o),$$

whenever $y \in A$, $y \not\equiv y_o$ satisfies $Gr(y) \subset S$ and (5.2.2). The last equality follows from the definition of J^* and the assumption that y_o is imbedded in $\mathcal{F}$.

2. Interpretation of Cavitation Using the Field Theory

We assume throughout this section the existence of a cavitating equilibrium solution $r_c \in C^2((0,1])$. We will also assume that the stored energy function Φ satisfies H1, H2, H3 and H5.

Remark 5.3

It follows from proposition 0.9 that r_c may be extended to $r_c \in C^2((0,\infty))$ as a solution of (0.2.3) satisfying $\frac{r_c(R)}{R} \searrow \lambda_c$ as $R \longrightarrow \infty$ for some $\lambda_c \in [1,\infty)$.

Proposition 5.4

Let

$$D_{\lambda_c} = \left\{ (R,r) \in \mathbb{R}^2 \; ; \; r > \lambda_c R \; , \; r > 0, \; R > 0 \right\} . \qquad (5.4.1)$$

Then

$$\mathcal{F}_c \overset{\text{def}}{=} \left\{ y(R,\partial) \; ; \; \partial \in (0,\infty) \right\} , \qquad (5.4.2)$$

where

$$y(R,\partial) \overset{\text{def}}{=} \partial \, r_c\left(\frac{R}{\partial}\right) \quad \text{for} \quad R \in (0,\infty) \; , \; \partial \in (0,\infty) \; , \qquad (5.4.3)$$

is a field of extremals of I over D_{λ_c} , where $r_c \in C^2((0,\infty))$ is a cavitating equilibrium solution with $\frac{r_c(R)}{R} \longrightarrow \lambda_c$ as $R \longrightarrow \infty$.

Proof

The set $\mathcal{F}_c$ consists of extremals because of the invariance of (0.2.3) under the scaling $(r,R) \longrightarrow (dr,dR)$ for $d \in (0,\infty)$. It follows from the properties of r_c that

$$y(R_0,\partial) \longrightarrow \lambda_c R_0 \quad \text{as} \quad \partial \longrightarrow 0$$

and

$$y(R_0,\partial) \longrightarrow \infty \quad \text{as} \quad \partial \longrightarrow \infty \; .$$

Hence there exists $\partial_0 \in (0,\infty)$ such that

$$y(R_0,\partial_0) = r_0 \; ;$$

so the extremals cover D_{λ_c}. The uniqueness of ∂_0 is a consequence of corollary 0.5.

Remark 5.5

Our assumption H1' implies that the Weierstrass excess function $\mathcal{E}$ corresponding to the integrand of (1.1.1) satisfies

$\xi(a,b,c,d) > 0$ for all $a,b,c,d \in (0,\infty)$ with $c \neq d$.

(see definition 2).

Our next proposition concerns the invertibility of the relation

$$v = \frac{r_c(R)}{R}.$$

Proposition 5.6

If $r_c \in C^2((0,\infty))$ is a cavitating equilibrium solution then there exists $g_c \in C^2((\lambda_c,\infty))$ satisfying

$$\text{(i)} \quad g_c\left(\frac{r_c(R)}{R}\right) = R \quad \text{for all } R \in (0,\infty), \tag{5.6.1}$$

$$\text{(ii)} \quad \lim_{v \to \infty} g_c(v) = 0, \tag{5.6.2}$$

$$\text{(iii)} \quad \lim_{v \to \lambda_c} g_c(v) = \infty. \tag{5.6.3}$$

Proof

The proof of this theorem is exactly analogous to that of proposition 3.7 and will be omitted.

For convenience we record the following analogue of proposition 3.8(i).

Proposition 5.7

If $r_c \in C^2((0,\infty))$ is a cavitating equilibrium solution and g_c is as defined in proposition 5.6, then

$$\frac{1}{v^3} H(v, r_c'(g_c(v))) \longrightarrow 0 \quad \text{as} \quad v \longrightarrow \infty, \tag{5.7.1}$$

where H is given by (0.7.3).

Proposition 5.8

Let $r_c \in C^2((0,\infty))$ be a cavitating equilibrium solution and let $P_c : D_{\lambda_c} \longrightarrow \mathbb{R}$ be defined by

$$P_c(R_o,r_o) = r_c'\left(g_c\left(\frac{r_o}{R_o}\right)\right) \quad \text{for } (R_o,r_o) \in D_{\lambda_c} \tag{5.8.1}$$

where g_c is as defined in proposition 5.6. Then $P_c \in C^1(D_{\lambda_c})$ and is the slope function corresponding to the field of extremals $\mathcal{F}_c$ given in proposition 5.4.

Proof

The proof follows from definition 5 and proposition 5.4.

We next prove that any cavitating solution r_c is a strict minimum of the energy amongst all functions $r \in A_\lambda$ with $Gr(r) \subset D_{\lambda_c}$.

Theorem 5.9

Let $r_c \in C^2((0,\infty))$ be a cavitating equilibrium solution with $\frac{r(R)}{R} \longrightarrow \lambda_c$ as $R \longrightarrow \infty$. If $r_c(1) = \lambda$ then

$$I(r_c) < I(r) \tag{5.9.1}$$

for all $r \in A_\lambda$ with $Gr(r) \subset D_{\lambda_c}$, $r \not\equiv r_c$.

Proof

It follows from proposition 5.4 that $\mathcal{F}_c$ defined by (5.4.2) is a field of extremals of I over D_{λ_c}. It is consequence of H1 that

$$R_o^2\Phi\left(q, \frac{r_o}{R_o}, \frac{r_o}{R_o}\right) > R_o^2\left[\Phi\left(P_c(R_o,r_o), \frac{r_o}{R_o}, \frac{r_o}{R_o}\right) + (q-P_c(R_o,r_o))\Phi_{,1}\left(P_c(R_o,r_o), \frac{r_o}{R_o}, \frac{r_o}{R_o}\right)\right]$$

$$\text{for all } (R_o,r_o) \in D_{\lambda_c},\quad q \in (0,\infty) \quad q \neq P_c(R_o,r_o), \tag{5.9.2}$$

where P_c is the slope function corresponding to the field $\mathcal{F}_c$ as defined by (5.8.1). Notice that (5.9.2) is the restatement of condition (5.2.1) of theorem 5.2 for the integrand $R^2\Phi$ with the field of extremals $\mathcal{F}_c$.

Now let $r \in A_\lambda$ satisfy $r \not\equiv r_c$, $Gr(r) \subset D_{\lambda_c}$ and $I(r) < +\infty$ (the result of the theorem holds trivially if $I(r) = +\infty$ by proposition 0.16).

Then using (5.9.2) we obtain

$$R^2\Phi(r', \tfrac{r}{R}, \tfrac{r}{R}) > R^2\left[\Phi(P_c(R,r), \tfrac{r}{R}, \tfrac{r}{R}) + (r'-P_c(R,r))\Phi_{,1}(P_c(R,r), \tfrac{r}{R}, \tfrac{r}{R})\right] \quad (5.9.3)$$

for a.e. $R \in (0,1)$.

Step 1

We claim that the right hand side of (5.9.3) is equal to

$$\frac{1}{3}\frac{d}{dR}\left[R^3 H\left(\frac{r(R)}{R}, P_c(R,r(R))\right)\right] \text{ for a.e. } R \in (0,1), \quad (5.9.4)$$

where H is defined by (0.7.3). The expression (5.9.4) is equal almost everywhere to

$$R^2H(\tfrac{r}{R},P_c(R,r)) + \frac{R^3}{3}\left[\Phi_{,1}\frac{d}{dR}P_c(R,r) + 2\Phi_{,2}\frac{1}{R}(r'-\tfrac{r}{R}) + (\tfrac{1}{R}(r'-\tfrac{r}{R}) - \tfrac{d}{dR}P_c(R,r))\Phi_{,1}\right.$$
$$\left. + (\tfrac{r}{R} - P_c(R,r))\frac{d}{dR}\Phi_{,1}\right] \quad (5.9.5)$$

$$= R^2\left[\Phi + (r'-P_c(R,r))\Phi_{,1}\right] + \frac{R^3}{3}\left[2(\Phi_{,2}-\Phi_{,1})\tfrac{1}{R}(r'-\tfrac{r}{R}) + (\tfrac{r}{R} - P_c(R,r))\frac{d}{dR}\Phi_{,1}\right]$$

(5.9.6)

To prove the claim it is therefore sufficient to show that the expression in square brackets in (5.9.6) is equal to zero almost everywhere i.e. that

$$(P_c(R,r)-\tfrac{r}{R})R\frac{d}{dR}\Phi_{,1}(P_c(R,r), \tfrac{r}{R}, \tfrac{r}{R}) = 2\left[\Phi_{,2}(P_c(R,r), \tfrac{r}{R}, \tfrac{r}{R}) - \right.$$
$$\left.\Phi_{,1}(P_c(R,r) \tfrac{r}{R}, \tfrac{r}{R})\right]\frac{1}{R}(r'-\tfrac{r}{R}) \quad (5.9.7)$$

for a.e. $R \in (0,1)$.

and setting $w = \frac{r}{R}$ in (5.9.7) this is equivalent to showing that

$$\left[r_c'(g_c(\tfrac{r}{R})) - \tfrac{r}{R})(r'-\tfrac{r}{R})\right]\frac{d}{dw}\Phi_{,1}(r_c'(g_c(w)),w,w)\Big|_{w=\frac{r}{R}} = 2\left[\Phi_{,2}(r_c'(g_c(\tfrac{r}{R})), \tfrac{r}{R}, \tfrac{r}{R}) - \right.$$
$$\left.\Phi_{,1}(r_c'(g_c(\tfrac{r}{R})), \tfrac{r}{R}, \tfrac{r}{R})\right].(r'-\tfrac{r}{R}), \quad (5.9.8)$$

for a.e. $R \in (0,1)$,

where we have used the definition of P_c (5.8.1). But r_c is a solution of (0.2.3) and setting $v = \frac{r_c}{R}$ gives

$$R \frac{d}{dR} = (r_c'(g_c(v))-v) \frac{d}{dv}. \tag{5.9.9}$$

Hence

$$(r_c'(g_c(v))-v) \frac{d}{dv} (\Phi,_1(r_c'(g_c(v)),v,v)) = 2[\Phi,_2(r_c'(g_c(v)),v,v) - \Phi,_1(r_c'(g_c(v)),v,v)]$$

for $v \in (\lambda_c,\infty)$.

Comparison with (5.1.8) then proves the claim.

Step 2

We next show that the right hand side of (5.9.3) is in $L^1(0,1)$. It follows from propositions 0.6, 5.6, corollary 0.5 and (0.3.6) that

$$0 \le \frac{1}{v^2} \Phi,_1(r_c'(g_c(v)),v,v) \le \frac{1}{\lambda_c^2} \Phi,_1(\lambda_c,\lambda_c,\lambda_c) \qquad v \in [\lambda_c,\infty) . \tag{5.9.10}$$

Since $Gr(r) \subset D_{\lambda_c}$ by assumption we conclude from the definition of P_c that $R^2\Phi(P_c(R,r), \frac{r}{R}, \frac{r}{R})$ is bounded. It follows from propositions 0.12 and 0.9(a) that r_c' is bounded and hence

$$(r'-P_c(R,r))R^2\Phi(P_c(R,r), \frac{r}{R}, \frac{r}{R}) \in L^1(0,1), \tag{5.9.11}$$

since $r' \in L^1(0,1)$ by assumption. As $I(r) < +\infty$; (5.9.11) and (5.9.13) imply that

$$R^2\Phi(P_c(R,r), \frac{r}{R}, \frac{r}{R}) \in L^1(0,1), \tag{5.9.12}$$

which together with (5.9.11) implies that the right hand side of (5.9.3) is in $L^1(0,1)$.

Step 3

It follows from Steps 1, 2, (5.9.3) and the fundamental theorem of calculus that

$$I(r) = \int_0^1 R^2\Phi(r', \tfrac{r}{R}, \tfrac{r}{R})dR \geq \frac{1}{3}\int_0^1 \frac{d}{dR}(R^3 H(\tfrac{r}{R}, P_c(R,r)))dR, \tag{5.9.13}$$

$$= \frac{1}{3}\left[H(\lambda, P_c(1,\lambda)) - \lim_{R \to 0} R^3 H(\tfrac{r}{R}, P_c(R,r))\right]. \tag{5.9.14}$$

We will show that (5.9.1) holds in the two cases

$$\text{(i)} \quad \overline{\lim_{R \to 0}} \frac{r(R)}{R} < +\infty, \tag{5.9.15}$$

$$\text{(ii)} \quad \overline{\lim_{R \to 0}} \frac{r(R)}{R} = +\infty. \tag{5.9.16}$$

If (i) holds then there exists a constant M such that

$$\lambda_c < \frac{r(R)}{R} \leq M \quad \text{for } R \text{ sufficiently small.} \tag{5.9.17}$$

Then setting $v = \frac{r(R)}{R}$ we obtain

$$|H(v, r_c'(g_c(v)))| \leq \text{Const} \quad \text{for } v \in [\lambda_c, M] \tag{5.9.18}$$

from continuity. Hence by (5.9.18) and (5.8.1)

$$\lim_{R \to 0} R^3 H(\tfrac{r}{R}, P_c(R,r)) = \lim_{R \to 0} R^3 H(\tfrac{r}{R}, r_c'(g_c(\tfrac{r}{R}))) = 0. \tag{5.9.19}$$

Inequality (5.9.1) is then a consequence of (5.9.13), (5.9.14), (5.9.19) and proposition 0.16(iii).

If case (ii) holds and $\varepsilon > 0$ it follows from proposition 5.7 that

$$|R^3 H(\tfrac{r}{R}, P_c(R,r))| < \varepsilon \quad \text{for all } R \text{ satisfying } \frac{r(R)}{R} \geq \tilde{M}$$

for some constant $\tilde{M}$. Then applying the arguments of case (i) on the interval $[\lambda_c, \tilde{M}]$ we again conclude that (5.9.19) and thus (5.9.1) holds.

As a corollary we have the following alternative proof of theorem 3.14.

<u>Theorem 5.10</u>

For each $\lambda \in [1, \infty)$ there exists at most one cavitating equilibrium solution $r_c \in C^2((0,1])$ satisfying $r_c(1) = \lambda$.

Proof

We suppose for a contradiction that there exist $\lambda_o \in (0,\infty)$ and $r_c, r_{\tilde{c}} \in C^2((0,1])$. distinct cavitating equilibrium solutions satisfying $r_c(1) = r_{\tilde{c}}(1) = \lambda_o$. Then by corollary 0.5 $Gr(r_{\tilde{c}}) \subset D_{\lambda_c}$ and applying theorem 5.9 we obtain $I(r_c) < I(r_{\tilde{c}})$. Reversing the roles of r_c and $r_{\tilde{c}}$ we obtain a contradiction.

Remark 5.11

Theorem 5.9 is a modified application of theorem 5.2, the main difference being that the endpoints of the admissable curves lie on the boundary of the region under consideration (theorem 5.2 circumvents this difficulty by assuming that they lie in the interior).

We next construct a field of extremals $\mathcal{F}$ over $\mathbb{R}^2_{++}$ by extending $\mathcal{F}_c$. Define by

$$\mathcal{F} = \mathcal{F}_c \cup \mathcal{F}_t \tag{5.11.1}$$

where $\mathcal{F}_c$ is as in proposition 5.4 and $\mathcal{F}_t$ is given by

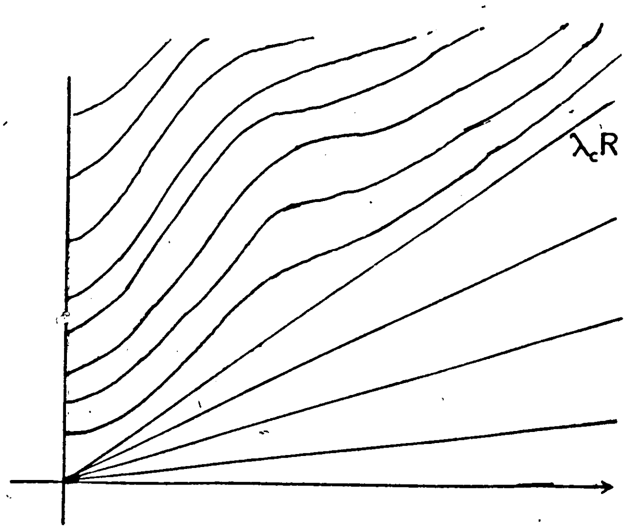
$$\mathcal{F}_t = \left\{ y(R,\partial) = (\lambda_c + \partial)R \, ; \, \partial \in (-\lambda_c, 0) \right\}. \tag{5.11.2}$$

Then the following is an easy consequence.

Proposition 5.12

$\mathcal{F}$ as defined by (5.11.1) and (5.11.2) is a field of extremals of I over $\mathbb{R}^2_{++}$.

The field $\mathcal{F}$



It is a consequence of proposition 5.8 and the definition of $\mathcal{F}_t$ that the slope function P corresponding to the field $\mathcal{F}$ is given by

$$P(R_o,r_o) = \begin{cases} P_c(R_o,r_o) & \text{if } r_o > \lambda_c R_o \qquad (5.12.1) \\ \dfrac{r_o}{R_o} & \text{if } r_o \leqslant \lambda_c R_o\,. \qquad (5.12.2) \end{cases}$$

Proposition 5.13

Let $r_c \in C^2((0,\infty))$ be a cavitating equilibrium solution and let $\mathcal{F}$ be defined by (5.11.1). Then for each $\lambda \in (0,\infty)$ and $r \in A_\mu$, $r \in \mathcal{F}$ with $I(r) < +\infty$ and $\underset{R \to 0}{\underline{\mathrm{Lim}}} \dfrac{r(R)}{R} > 0$. the Hilbert integral I^* satisfies

$$I(r) > I^*(r) = \left[H(\mu,P(1,\mu))\right] - \underset{R \to 0}{\mathrm{Lim}} \left[R^3 H(\tfrac{r(R)}{R},P(R,r(R)))\right], \qquad (5.13.1)$$

where P is given by (5.12.1), (5.12.2) and H by (0.7.3).

Proof

Let r satisfy the hypotheses of the proposition; we first show that

$$\int_a^b R^2\left[\Phi(P(R,r),\tfrac{r}{R},\tfrac{r}{R}) + (P(R,r)-r')\Phi_{,1}(P(R,r),\tfrac{r}{R},\tfrac{r}{R})\right]dR = \frac{1}{3}\Big|_a^b R^3 H(R,P(R,r))$$

whenever $0 < a < b \leqslant 1$ and $\dfrac{r(R)}{R} \neq \lambda_c$ for $R \in (a,b)$. (5.13.1)

It follows from Step 1 of the proof of theorem 5.9, proposition 0.9(b) and (5.12.1) that (5.13.1) holds on any interval for which $\dfrac{r(R)}{R} > \lambda_c$ for $R \in (a,b)$. To verify that it holds in the case where $\dfrac{r(R)}{R} < \lambda_c$ for $R \in (a,b)$; we note that by (5.12.2)

$$\frac{1}{3}\frac{d}{dR}\left[R^3 H(R,P(R,r))\right] = \frac{1}{3}\frac{d}{dR}\left[R^3\Phi(\tfrac{r}{R},\tfrac{r}{R},\tfrac{r}{R})\right]$$

$$= R^2\left[\Phi(\tfrac{r}{R},\tfrac{r}{R},\tfrac{r}{R}) + (\tfrac{r}{R}-r')\Phi_{,1}(\tfrac{r}{R},\tfrac{r}{R},\tfrac{r}{R})\right]$$

and so (5.14.1) clearly holds.

The arguments of Step 2 in the proof of theorem 5.9 together with our assumption that $\underline{\mathrm{Lim}}_{R \to 0} \frac{r(R)}{R} > 0$ then imply that $R^3 H(R,P(R,r)) \in W^{1,1}(0,1)$, proving the proposition.

Using the results of proposition 5.13 we obtain the following theorem.

<u>Theorem 5.14</u>

Let $r_c \in C^2((0,\infty))$ be a cavitating equilibrium solution and let $\mathcal{F}$ be defined by (5.11.1). Then for each $\lambda \in (0,\infty)$ if $y \in \mathcal{F}$ is the unique element satisfying $y(1) = \lambda$ then

$$I(y) < I(r)$$

for all $r \in A_\lambda$, $r \not\equiv y$, with $I(r) < +\infty$ and $\underline{\mathrm{Lim}}_{R \to 0} \frac{r(R)}{R} > 0$.

<u>Proof</u>

The proof is an easy consequence of proposition 5.13 on noting that firstly the arguments of Step 3 in the proof of theorem 5.9 and the assumption that $\underline{\mathrm{Lim}}_{R \to 0} \frac{r(R)}{R} > 0$ together imply that $\mathrm{Lim}_{R \to 0} \; R^3 H(R,P(R,r(R))) = 0$ and secondly that $I^*(z) = I(z)$ for all $z \in \mathcal{F}$. Thus

$$I(r) > I^*(r) = I^*(y) = I(y).$$

Theorem 5.14 is another modified application of theorem 5.2.

Finally we indicate how the assumption $\underline{\mathrm{Lim}}_{R \to 0} \frac{r(R)}{R} > 0$ may be relaxed.

To do this we first note that the conclusions of theorem 5.14 hold if

$R_n^3 H(R_n, P(R_n,r(R_n))) \longrightarrow 0$ as $n \longrightarrow \infty$ for some sequence $R_n \longrightarrow 0$.

If such a sequence did not exist then $\mathrm{Lim}_{R \to 0} \frac{r(R)}{R} = 0$ and this would imply the existence of a constant $k > 0$ such that

$$R^3\Phi\left(\frac{r}{R}, \frac{r}{R}, \frac{r}{R}\right) \geq K \quad \text{for} \quad R \in (0,1] . \qquad (5.14.1)$$

We now assume that Φ satisfies the further condition

$$\Phi(v_1,v_2,v_3) = \sum_{i=1}^{3} \psi(v_i) + \tilde{\Phi}(v_1,v_2,v_3) , \qquad (5.14.2)$$

where

$$\tilde{\Phi},\ \psi > 0 \text{ and } \overline{\lim_{v \to 0^+}}\ \tilde{\Phi}(v,v,v) < +\infty \qquad (5.14.3)$$

Thus if $r \in A_\lambda$ satisfies $I(r) < +\infty$ it follows from (5.14.2) that $R^2\psi(\frac{r}{R}) \in L^1(0,1)$. But by (5.14.1) - (5.14.3)

$$0 < \frac{K}{R} \leq R^2\Phi\left(\frac{r}{R}, \frac{r}{R}, \frac{r}{R}\right) \leq 3R^2\psi\left(\frac{r}{R}\right) + \text{const} ,$$

contradicting the fact that $R^2\psi(\frac{r}{R}) \in L^1(0,1)$. Thus for stored energy functions of the structure indicated if $r \in A_\lambda$ and $I(r) < +\infty$ then $\underline{\lim}_{R \to 0} \frac{r}{R} > 0$.

In closing we remark that the results of this chapter indicate that higher dimensional field theories will be useful tools in tackling problems in non linear elasticity.

APPENDIX

Constitutive Assumptions

Throughout this thesis unless otherwise stated we assume that

$\Phi \in C^3(\mathbb{R}^3_{++})$ and that $\Phi,_i(1,1,1) = 0$ so that the undeformed configuration is a natural state. We will also refer to the following hypotheses on Φ.

(H1) $$\Phi,_{11}(v_1,v_2,v_3) > 0$$

This is known as the tension-extension inequality. For an interpretation of H1 see Truesdell and Noll (1965).

(H2) $$\left[\frac{v_i\Phi,_i(v_1,v_2,v_3)-v_j\Phi,_j(v_1,v_2,v_3)}{v_i - v_j}\right] \geq 0 \;, \quad i \neq j \;, \quad v_i \neq v_j$$

We say that Φ satisfies H2$^+$ if strict inequality holds. The set of inequalities H2$^+$ are known as the Baker-Ericksen inequalities (see Truesdell and Noll for an interpretation).

(H3) Either

$$\lim_{\substack{v_1,v_2 \to \\ v_1 < v_2}} \left[\frac{\Phi,_1(v_1,v_2,v_3)}{v_2^2}\right] = \infty$$

or

$$\lim_{\substack{v_1,v_2 \to 0 \\ v_1 < v_2}} \left[\Phi(v_1,v_2,v_2) - v_1\Phi,_1(v_1,v_2,v_2)\right] = -\infty$$

(H4) Either

$$\lim_{\substack{v_1,v_2 \to 0 \\ v_1 > v_2}} \left[\frac{\Phi,_1(v_1,v_2,v_2)}{v_2^2}\right] = -\infty$$

or

$$\lim_{\substack{v_1, v_2 \to 0 \\ v_1 > v_2}} \left[\Phi(v_1,v_2,v_2) - v_1\Phi_{,1}(v_1,v_2,v_2) \right] = +\infty$$

(H5) $\Phi_{,1}(v,a,a) \longrightarrow +\infty$ (respectively $-\infty$) as $v \longrightarrow \infty$ (respectively 0)

for fixed $a \in (0,\infty)$.

(H6) $$\det(\text{Hess } \Phi)\Big|_{v_i=1} \overset{\text{def}}{=} \det(\Phi_{,ij}(1,1,1)) > 0.$$

(H7) $$\frac{\Phi_{,i} - \Phi_{,j}}{v_i - v_j} + \Phi_{,ij} \geqslant 0.$$

(H8) $\Phi_{,i}(v,v,v) = 0$ for all i if and only if $v = 1$.

This is the assumption that Φ has only one natural state.

(H9) $$\frac{v^2}{(v^3-1)^2}\,\hat{\Phi}(v) \in L^1(\partial,\infty) \quad \text{for } \partial \in [1,\infty] \text{ where}$$

$$\hat{\Phi}(v) \overset{\text{def}}{=} \Phi\left(\frac{1}{v^2},v,v\right).$$

(H10) There exist constants $M > 0, k$ such that

$$M \leqslant \frac{\Phi(\lambda,\lambda,\lambda)}{\lambda^3} \quad \text{for } \lambda \geqslant k$$

(H11) $\Phi(v_1,v_2,v_3) > \Phi(1,1,1)$ if $v_i \neq 1$ for some i.

(E1) $$\Phi(v_1,v_2,v_3) \geqslant \sum_{i=1}^{3} \Psi(v_i) \quad \text{where}$$

$\Psi : (0,\infty) \longrightarrow (0,\infty)$ satisfies

(i) $\Psi \in C((0,\infty))$

(ii) $\dfrac{\Psi(v)}{v} \longrightarrow \infty$ as $v \longrightarrow \infty$

(iii) $\Psi(v) \longrightarrow \infty$ as $v \longrightarrow 0$.

(E2) $\left|\Phi_{,i}(v_1,\alpha_2 v_2,\alpha_3 v_3)v_i\right| < M(\Phi(v_1,v_2,v_3)+1)$ if $|\alpha_i - 1| < \varepsilon_o$, $i = 2,3$

for some constants $M, \varepsilon_o \in (0,\infty)$.

Further Constitutive Assumptions for Chapter 2

The function $f : \mathbb{R}^+ \times (-1,\infty) \longrightarrow \mathbb{R}^+$ satisfies f1 if there exists a constant $k_o \in (0,\infty)$ such that

(i) $f(k,\cdot) \in C^3((-1,\infty))$ and is convex for each $k \in (0,k_o)$,

(ii) $f'(k,v) \longrightarrow \infty$ as $v \longrightarrow \infty$ for each $k \in (0,k_o)$,

(iii) $f(k,0) \longrightarrow c$ as $k \longrightarrow 0$, where $c \in [0,\infty)$ is a constant,

(iv) $c_1\frac{|v|}{k} + c_2 \leqslant f(k,v)$ for $k \in (0,k_o)$ where $c_1 > 0$ and $c_2 \geqslant 0$ are constants,

(v) $f(k,v) \longrightarrow \infty$ as $v \longrightarrow -1$ from above for each $k \in (0,k_o)$,

(vi) $\left|f'(k,v)\right| < M(\frac{f(k,v)}{v+1} + 1)$ for $k \in (0,k_o)$ and $v \in (-1,\infty)$, where $M \in (0,\infty)$ is a constant.

(Φ1) $$\Phi(v_1,v_2,v_3) \geqslant \sum_{i=1}^{3} \psi(v_i) \text{ where } \psi : [0,\infty) \longrightarrow [0,\infty)$$

satisfies

(i) ψ is continuous

(ii) $\frac{\psi(v)}{v} \longrightarrow \infty$ as $v \longrightarrow \infty$.

(Φ2) $$\overline{\lim_{v \to 0^+}} \Phi_{,1}(v,a,a) < +\infty \quad \text{for } a \in (0,\infty)$$

and

$$\underline{\lim_{v \to \infty}} \Phi_{,1}(v,a,a) > -\infty \quad \text{for } a \in (0,\infty).$$

(Φ3) $$\frac{1}{v^3-1}\frac{d\hat{\Phi}(v)}{dv} \in L^1(1,\infty) \text{ where } \hat{\Phi}(v) \overset{\text{def}}{=} \Phi(\tfrac{1}{v^2},v,v).$$

(Φ4) There exist constants $A,B > 0$ and $\beta \in (0,2)$ such that

$$\left[\frac{v_1\Phi_{,1}(v_1,v_2,v_2) - v_2\Phi_{,2}(v_1,v_2,v_2)}{v_1-v_2}\right] \leq A + B(v_2)^{\beta}$$

for $0 < v_1 \leq v_2$.

Proof of Proposition 0.3

The proof uses a technique from Ball (1982) theorem 7.3

Let $k \in (1,\infty)$ and define s_k by

$$s_k = \left\{\rho \in (\tfrac{1}{k},1) \ ; \ \tfrac{1}{k} < r'(\rho) < k\right\}. \tag{6.0.1}$$

let $v \in L\,(0,1)$ satisfy

$$\int_{s_k} v\,d\rho = 0. \tag{6.0.2}$$

Then setting

$$r_\varepsilon(\rho) = r(\rho) + \varepsilon.\int_0^\rho v(\tau)x_k(\tau)d\tau, \tag{6.0.3}$$

where x_k is the characteristic function of s_k, it follows from (6.0.2) and (6.0.3) that r_ε satisfies

(i) $r_\varepsilon(1) = \lambda$,

(ii) $r_\varepsilon(0) = r(0)$,

(iii) $r_\varepsilon'(\rho) = r'(\rho)$ if $\rho \leq \frac{1}{k}$ or if $r'(\rho) \notin (\frac{1}{k},k)$.

Since $r \in C([\frac{1}{k},1])$ and $r' > 0$ a.e., $r(\frac{1}{k}) > 0$ and so $r_\varepsilon(\rho) > 0$ for $\rho \in (0,1)$ provided ε is sufficiently small. It follows from (iii) that $r_\varepsilon'(\rho) > 0$ for a.e. $\rho \in (0,1)$ provided $\varepsilon < \dfrac{1}{2k\,.\,\|v\|_\infty}$ and thus

$r_\varepsilon \in A_\lambda$ for sufficiently small ε. The triangle inequality implies that

$$\left|\frac{\Phi(r_\varepsilon',\frac{r_\varepsilon}{\rho},\frac{r_\varepsilon}{\rho})-\Phi(r',\frac{r}{\rho},\frac{r}{\rho})}{\varepsilon}\right| \leq \left|\frac{\Phi(r_\varepsilon',\frac{r_\varepsilon}{\rho},\frac{r_\varepsilon}{\rho})-\Phi(r',\frac{r_\varepsilon}{\rho},\frac{r_\varepsilon}{\rho})}{\varepsilon}\right| + \left|\frac{\Phi(r',\frac{r_\varepsilon}{\rho},\frac{r_\varepsilon}{\rho})-\Phi(r',\frac{r}{\rho},\frac{r}{\rho})}{\varepsilon}\right|$$

for $\rho \in (0,1)$.

Notice that the above inequality is identically zero for $\rho \in (0, \frac{1}{k})$. If $\rho \in (\frac{1}{k}, 1)$ and $r'(\rho) \in (\frac{1}{k},k)$ then the two terms on the right hand side of (6.0.4) are bounded by a constant independent of ε. If $\rho \in (\frac{1}{k}, 1)$ and $r'(\rho) \notin (\frac{1}{k}, k)$ then, on multiplying by ρ^2, the right hand side of (6.0.4) takes the form

$$2\rho^2 . \left|\Phi_{,2}(r'(\rho),g(\rho,\theta(\rho),\varepsilon),g(\rho,\theta(\rho),\varepsilon))\right| . \frac{1}{\rho}\int_0^\rho v(\tau)x_k(\tau)d\tau \tag{6.0.5}$$

by (iii) and the mean value theorem, where

$$g(\rho,\theta(\rho),\varepsilon) = \frac{r(\rho)}{\rho} + \frac{\varepsilon\theta(\rho)}{\rho} . \int_0^\rho v(\tau)x_k(\tau)d\tau \tag{6.0.6}$$

and $\theta(\rho) \in (0,1)$.

We now write

$$g(\rho,\theta,\varepsilon) = \frac{r}{\rho} . \left[\frac{r(\rho) + \varepsilon\theta(\rho).\int_0^\rho x_k v d\tau}{r(\rho)}\right] \tag{6.0.7}$$

and using the fact that

$$\left|\frac{r(\rho) + \varepsilon\theta(\rho).\int_0^\rho x_k v d\tau}{r(\rho)} - 1\right| < \varepsilon_0 \tag{6.0.8}$$

for ε sufficiently small, we conclude from E2 that the right hand side of (6.0.4) is bounded by

$$2\rho^2 M.\left[\Phi(r',\tfrac{r}{R},\tfrac{r}{R}) + 1\right]\cdot \frac{1}{\rho}\int_0^\rho x_k v d\tau \cdot \left[\frac{\rho}{r(\rho)}\right] . \qquad (6.0.9)$$

Since $r(\rho) > r(\frac{1}{k})$ for $\rho \in (\frac{1}{k}, 1)$ and since $I(r) < +\infty$ by assumption it follows that (6.0.9) lies in $L^1(\frac{1}{k}, 1)$. As r is a global minimiser of I, on using the dominated convergence theorem we obtain

$$0 = \lim_{\varepsilon \to 0} \int_0^1 R^2 \left[\frac{\Phi(r_\varepsilon', \frac{r_\varepsilon}{R}, \frac{r_\varepsilon}{R}) - \Phi(r', \frac{r}{R}, \frac{r}{R})}{\varepsilon}\right] dR = \int_{\frac{1}{k}}^{1} R^2\Big[\Phi_{,1}(R)x_k(R)v(R) + 2\Phi_{,2}(R)\frac{1}{R}\cdot$$

$$\int_0^R x_k(\tau)v(\tau)d\tau\Big]\, dR. \qquad (6.0.10)$$

Since $I(r) < +\infty$, $r(\frac{1}{k}) > 0$, it follows from E2 that $R\Phi_2(r', \frac{r}{R}, \frac{r}{R}) \in L^1(\frac{1}{k}, 1)$

and integrating (6.0.10) by parts then gives

$$\int_{s_k} \left[R^2\Phi_{,1}(r', \tfrac{r}{R}, \tfrac{r}{R}) + 2\int_R^1 \rho\Phi_{,2}(r', \tfrac{r}{\rho}, \tfrac{r}{\rho})d\rho\right] . v(R)dR = 0. \qquad (6.0.11)$$

As (6.0.11) holds for all $v \in L^\infty(0,1)$ with $\int_{s_k} v d\rho = 0$ it follows that

$$R^2\Phi_{,1}(R) + 2\int_R^1 \rho\Phi_{,2}(\rho)d\rho = c_k \quad \text{for a.e. } \rho \in s_k ,$$

where c_k is a constant. Finally since $\operatorname{meas}\left((0,1)/\bigcup_1^\infty s_k\right) = 0$, the c_k are all equal and an application of Ball (1982) theorem 4.2 implies $r \in C^m((0,1])$ and satisfies (0.2.3) and (0.3.1).

If $r(0) > 0$ then let $w \in C^\infty((0,1))$ satisfy $w(\rho) = 1$ for $\rho \in (0, \frac{1}{3})$ and $w(\rho) = 0$ for $\rho \in (\frac{2}{3}, 1)$. It is a consequence of E2 that $R\Phi_{,2}(R) \in L^1(0,1$

On setting

$$u_\varepsilon(\rho) = r(\rho) + \varepsilon w(\rho)$$

we obtain

$$0 = \frac{d}{d\varepsilon}\left[I(u_\varepsilon)\right]\Bigg|_{\varepsilon=0} = \int_0^1 R^2\Phi_{,1}(R)w'(R)+2R\Phi_{,2}(R)w(R)\,dR = -\lim_{R\to 0} R^2\Phi_{,1}(R),$$

proving the proposition.

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